HOW INTEREST RATE CHANGES AFFECT BOND PRICES

The purpose of this note is to give a reasonably detailed explanation of the inverse relationship between a change in the general interest rate and the price of an existing bond. While the overall principle should become obvious after giving the matter any amount of thought, it is the precise, quantitative change in a bond's price that interests us. In order to understand how that works, we examine the technique of discounted cash flow analysis and its underlying rationale.

1. Cash flow, price and discounting

Let t_0 denote an initial point in time and fix once and for all a positive time interval Δt. For each positive integer i we define

$$t_i = t_0 + i\Delta t.$$

In this way we obtain an infinite sequence of equally-spaced times

$$t_0,\ t_1,\ t_2,\ t_3,\ \ldots$$

We are interested in financial products through which the seller agrees to pay the buyer a specified dollar amount C_i at each time t_i. The C_i are called the product's *cash flows*; its *total cash flow* is their sum. We focus our attention on the following two examples.

Product A. For the price of one dollar, this product pays out a single cash flow of $1 + r$ dollars at time t_1. A familiar example of this would be an interest-bearing bank account with interest rate r and compounding period Δt. As in that setting, we assume that such a product is available for purchase at any time t_i and that any non-negative quantity may be purchased. The perpetual availability of the product gives a person the ability to have their money invested in it over any time interval by immediately reinvesting the cash flow after every single interval Δt. For example, if one invested P dollars between times t_i and t_j with $i \leq j$, then the total cash flow would be $P(1 + r)^{j-i}$.

Product B. An arbitrary product sold at time t_0, terminating at time t_n for some $n \geq 1$ and having cash flows $C_1, \ldots, C_n$.

We now enter into a meditation on the price P one should expect to pay for Product B at time t_0. Whatever P is, we know that a potential buyer of Product B has the option to invest it in Product A instead. The future value of P at time t_n under the latter product is $P(1 + r)^n$. In order for Product B to be competitive, it needs to have that same future value. The subtlety is that when a *smart* owner of Product B receives the cash flow C_i at time t_i, they will immediately turn around and reinvest it in Product A. The future value of those reinvestments at time t_n will be

$$C_1(1+r)^{n-1} + C_2(1+r)^{n-2} + \cdots + C_n(1+r)^0.$$

Equating this with the future value $P(1+r)^n$ provided by Product A gives us

$$\boxed{P = \sum_{i=1}^{n} \frac{C_i}{(1+r)^i}.}$$

This is called the *discounted cash flow formula.* Using it to determine a product's price is called *discounting,* and in this context r is called the *discount factor.*

2. Bonds

A (*coupon*) *bond* is a financial product having the following terms.

- An *issue date* t_0.
- A *maturity date* t_n.
- A *par* (or *face*) *value* P_0, given in dollars.
- A *coupon rate* r_0.
- Cash flows $C_1, \dots, C_n$ given by

$$C_i = \begin{cases} r_0 P_0 & \text{if } 1 \le i \le n-1, \\ r_0 P_0 + P_0 & \text{if } i = n. \end{cases}$$

The amount $r_0 P_0$ paid out every period is called the *coupon.*

Example 1. Consider (the imaginary) Bond X, which is a ten-year bond that was issued on January 1, 2010 with par value of \$1,000 and coupon rate of 5%. Coupons are usually paid biannually, but for the sake of simplicity we will assume that the coupons of Bond X are paid annually. The purchaser of Bond X should therefore expect to receive \$50 every year up to and including January 1, 2020, at which time they should expect an additional payment of \$1,000.

Note that the terms of a bond mention nothing about its actual price. For a bond, the discounted cash flow formula may be written as

$$\begin{aligned} P &= \frac{1}{(1+r)^n}\left[r_0P_0(1+r)^{n-1} + \cdots + r_0P_0(1+r)^1 + (r_0P_0 + P_0)(1+r)^0\right] \\ &= \frac{P_0}{(1+r)^n}\left(1 + r_0\sum_{i=0}^{n-1}(1+r)^i\right) \\ &= \frac{P_0}{(1+r)^n}\left(1 + r_0\frac{1-(1+r)^n}{1-(1+r)}\right) \\ &= \frac{P_0}{(1+r)^n}\left(1 + r_0\frac{(1+r)^n - 1}{r}\right). \end{aligned}$$

Cleaning this up a bit, we obtain

$$\boxed{P = P_0\left(\frac{r_0}{r} + \frac{1}{(1+r)^n}\left(1 - \frac{r_0}{r}\right)\right).}$$

In the context of bonds, the discount rate r is called the *rate to maturity.*

An immediate consequence is the following.

Proposition. *A bond's price equals its par value if and only if its yield to maturity equals its coupon rate.*

Proof. This is confirmed by substituting $P = P_0$ or $r = r_0$ into the above formula. Alternatively, one might notice that the reinvestment of a bond's cash flows at the coupon rate is equivalent to the continuous investment of its par value at that same rate. □

We are now ready to price some bonds! The discounted cash flow formula describes the situation in which a person has the following options.

A. Invest capital P at the rate of maturity r over the life of the bond.
B. Use P to purchase the bond, receive its cash flows and reinvest them at rate r.

The rate r at which either program may be executed in real life is, of course, the prevailing interest rate. This suggests the following rule for computing a bond's price.

Set the yield to maturity in the discounted cash flow formula to the current interest rate.

Example 2. Continuing our discussion of Bond X from Example 1, suppose that, shortly after the issue date, the general interest rate fell to 4.5%. The price of Bond X would increase to

$$P = \$1,000\left(\frac{0.05}{0.045} + \frac{1}{1.045^{10}}\left(1 - \frac{0.05}{0.045}\right)\right) \approx \$1{,}040.$$

The thing that makes you laugh will also make you cry. Suppose instead that the interest rate rose to 5.5%. The price of Bond X would then decrease to

$$P = \$1,000\left(\frac{0.05}{0.055} + \frac{1}{1.055^{10}}\left(1 - \frac{0.05}{0.055}\right)\right) \approx \$962.$$

This demonstrates the inverse relationship between changes in interest rates and those in bond prices. It also exhibits the slight asymmetry in the effect of interest rate reductions on bond prices verses that of interest rate hikes.

We now present another application of the discounted cash flow formula to the analysis of bond prices. It might happen that the general interest rate experiences no change after a bond is issued, but for whatever reason its price is either above the par value (*selling at a premium*) or below the par value (*selling at discount*). This could happen, for example, if the ability of the issuer to repay the bond is perceived by the market to have improved or worsened due to circumstances within the business.

An investor thinking about purchasing such a bond will want to determine how advantaged or disadvantaged they would be due to the discrepancy in price. One question they could ask is the following.

What (virtual) interest rate would place the bond at its current price, provided it was originally selling at par?

This is inverse to asking the bond's price, so its answer is the bond's yield to maturity.

Example 3. If Bond X ended up selling above par at \$1,100 shortly after its issue, then its yield to maturity would be given by

$$\$1{,}100 = \$1{,}000\left(\frac{0.05}{r} + \frac{1}{(1+r)^n}\left(1 - \frac{0.05}{r}\right)\right).$$

There is no closed formula that allows us to solve this equation directly for r, but a bit of trial and error reveals that 3.78% is the solution.

On the other hand, if Bond X ended up selling below par at \$900 shortly after its issue, then its yield to maturity would be given by

$$\$900 = \$1{,}000\left(\frac{0.05}{r} + \frac{1}{(1+r)^n}\left(1 - \frac{0.05}{r}\right)\right).$$

The solution to this equation is 6.38%.