PRESHEAVES ARE COLIMITS OF REPRESENTABLE FUNCTORS

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Let Set denote the category of all sets. The following is well known.

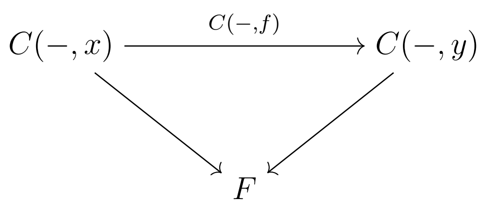
Theorem. *If C is a category, then every contravariant functor F from C to Set (presheaf) is a colimit of representable functors.*

In this note we give a detailed account of what this means and why it is true, providing two proofs, the first previously unknown to the author. We caution readers who are averse to Yoneda's lemma to turn back at this point.

The setup. Let $Y\colon C \to [C^{\mathrm{op}}, \mathsf{Set}]$ denote the Yoneda embedding. Recall that objects in the slice category $(Y \downarrow F)$ are natural transformations

$$C(-,x) \longrightarrow F\ , \qquad x \in C.$$

A morphism from $C(-,x) \to F$ to $C(-,y) \to F$ in $(Y \downarrow F)$ is a morphism $f\colon x \to y$ in C that makes the diagram



commute. It turns out that $(Y \downarrow F)$ is the diagram category over which F is a colimit. For the sake of brevity we will write $Y(x)$ and $Y(f)$ instead of $C(-,x)$ and $C(-,f)$ throughout the remainder of our discussion.

Let $P\colon (Y \downarrow F) \to C$ denote the 'forgetful functor' that assigns an object $Y(x) \to F$ to x and a morphism f to itself. We construct a cone from $Y \circ P$ to F by assigning to each object $\phi\colon Y(x) \to F$ in $(Y \downarrow F)$ the projection

$$\phi\colon (Y \circ P)(\phi) = Y(x) \longrightarrow F$$

in $[C^{\mathrm{op}}, \mathsf{Set}]$. That these projections form a cone follows automatically from the definition of $(Y \downarrow F)$, as illustrated by the commutativity of the above diagram. To prove that F is a colimit, we need to show that this cone is initial.

A ‘natural’ approach. Select an arbitrary cone from $Y \circ P$ to a functor $G \in [C^{\mathrm{op}}, \mathsf{Set}]$. For each $\phi \in (Y \downarrow F)$, let π_ϕ denote the corresponding projection in the cone over G. The key observation is that an assignment of morphisms

$$(\phi\colon Y(x) \to F) \longmapsto (\pi_\phi\colon Y(x) \to G)$$

defines a cone from $Y \circ P$ to G if and only if

$$\pi\colon\ \mathrm{Nat}(Y(-), F) \longrightarrow \mathrm{Nat}(Y(-), G)$$

is a natural transformation of functors from C^{op} to Set. (Here we have abused notation a bit, identifying π_ϕ and $\pi_x(\phi)$.)

Now consider the evaluation bifunctor $E\colon C \times [C^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{Set}$ defined on objects by

$$E(x, H) = H(x).$$

Yoneda’s lemma provides a natural isomorphism of bifunctors

$$\iota\colon\ \mathrm{Nat}(Y(-), -) \xrightarrow{\ \sim\ } E.$$

Since ι is a natural isomorphism, there exists a unique natural transformation $\theta\colon F \to G$ that makes the following diagram commute for all $x \in C$.

$$\begin{array}{ccc} \mathrm{Nat}(Y(x), F) & \xrightarrow{\iota_{(x,F)}} & F(x) \\ {\scriptstyle \pi_x}\big\downarrow & & \big\downarrow{\scriptstyle \theta_x} \\ \mathrm{Nat}(Y(x), G) & \xrightarrow[\iota_{(x,G)}]{} & G(x) \end{array}$$

Then θ is precisely the cone morphism required to show that our original cone from $Y \circ P$ to F is initial. (It is straightforward to check that completing the above diagram is equivalent to being a cone morphism, by applying $\iota_{(x,G)}$ to the definition.)

The entire proof may be further condensed to the following statement.

Proof. The category of cones under $Y \circ P$ is equivalent to the category of elements of the functor $K\colon [C^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{Set}$ given by

$$K(G) = \mathrm{Nat}(\mathrm{Nat}(Y(-), F), \mathrm{Nat}(Y(-), G)).$$

Moreover, conjugation by ι induces a natural isomorphism of functors

$$\tilde{\iota}\colon K \xrightarrow{\ \sim\ } \mathrm{Nat}(F, -)$$

under which a cone $(G, \pi) \in \mathsf{el}\, K$ becomes identified with the unique cone morphism from F to G. $\square$

For those left feeling drowned, rather than purified, by this baptism of category theory, we now recite the traditional proof in gory detail.

The pedestrian route. Select an arbitrary cone from $Y \circ P$ to a functor $G \in [C^{\text{op}}, \mathsf{Set}]$. For each $\phi \in (Y \downarrow F)$, let π_ϕ denote the corresponding projection in the cone over G. To construct the required natural transformation $\theta\colon F \to G$ on objects, let $x \in C$. In light of the Yoneda isomorphism

$$\iota\colon \operatorname{Nat}(Y(x), F) \xrightarrow{\sim} F(x),$$

every element of $F(x)$ may be written uniquely as $\iota(\phi)$ for some $\phi\colon Y(x) \to F$. (Note that we have abbreviated the notation $\iota_{(x,F)}$ of the previous section.) We set

$$\theta_x(\iota(\phi)) = \iota(\pi_\phi) \in G(x),$$

where

$$\iota\colon \operatorname{Nat}(Y(x), G) \xrightarrow{\sim} G(x)$$

again denotes the Yoneda isomorphism.

To prove that θ is a natural transformation, we need to show that the diagram

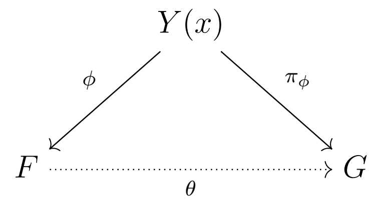
$$\begin{array}{ccc} F(y) & \xrightarrow{\theta_y} & G(y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(x) & \xrightarrow[\theta_x]{} & G(x) \end{array}$$

commutes for each morphism $f\colon x \to y$ in C. So let $\iota(\psi) \in F(y)$ with $\psi\colon Y(y) \to F$. For convenience, write $\phi = \psi \circ Y(f)$. In the cone over G we will then have $\pi_\phi = \pi_\psi \circ Y(f)$. It follows that

$$\begin{aligned}
(G(f) \circ \theta_y)(\iota(\psi)) &= (G(f) \circ \iota)(\pi_\psi) \\
&= \iota(\pi_\psi \circ Y(f)) && \text{by the naturality of } \iota \\
&= \iota(\pi_\phi) \\
&= \theta_x(\iota(\phi)) \\
&= \theta_x(\iota(\psi \circ Y(f))) \\
&= \theta_x((F(f) \circ \iota)(\psi)) && \text{by the naturality of } \iota \\
&= (\theta_x \circ F(f))(\iota(\psi)),
\end{aligned}$$

so the above diagram does commute.

To prove that θ is a morphism of cones, we need to check that every diagram



commutes. By the way in which the Yoneda isomorphism is defined, we have

$$\iota(\theta \circ \phi) = (\theta \circ \phi)_x(1_x) = \theta_x(\phi_x(1_x)) = \theta_x(\iota(\phi)) = \iota(\pi_\phi).$$

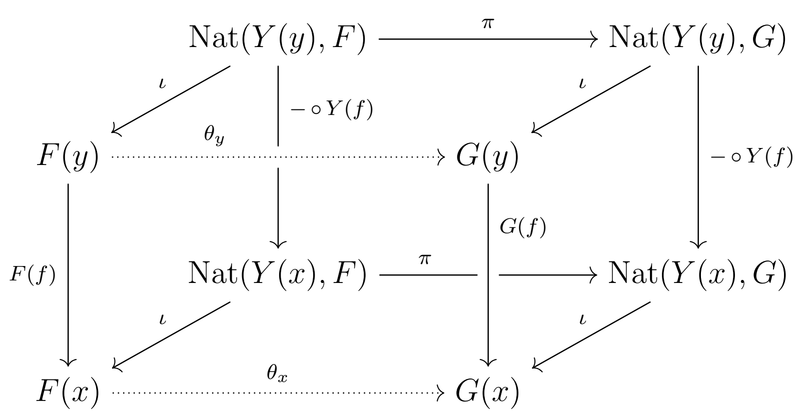
The injectivity of ι now forces $\theta \circ \phi = \pi_\phi$.

Finally, to prove that θ is unique we select an arbitrary cone morphism $\tau\colon F \to G$. For any $x \in C$ and $\iota(\phi) \in F(x)$ we then have

$$\begin{aligned} \tau_x(\iota(\phi)) &= (\tau_x \circ \iota)(\phi) \\ &= \iota(\tau \circ \phi) && \text{by the naturality of } \iota \\ &= \iota(\pi_\phi) && \text{since } \tau \text{ is a cone morphism} \\ &= \theta_x(\iota(\phi)). \end{aligned}$$

This shows that $\tau_x = \theta_x$ for all $x \in C$, hence $\tau = \theta$.

Remark. Every detail in the above argument may also be read off of the cube diagram



whose solid edges are known to commute.