ASSIGNMENT 2, DUE MONDAY 25 SEPTEMBER

1. Use the row reduction algorithm to compute the solution set of the system

$$\begin{aligned} \overline{5}x_1 + \overline{2}x_2 &= \overline{10} \\ \overline{2}x_1 + \overline{3}x_2 + x_3 &= \overline{9} \\ \overline{10}x_1 + \overline{4}x_2 &= \overline{9} \end{aligned}$$

over the field $\mathbb{Z}/11$.

2. Let F be a field. Determine whether or not the following are vector spaces over F. Explain your reasoning.

(a) The set of all polynomials over F that are divisible by the monomial x.

(b) The set of all polynomials over F having even degree, along with the zero polynomial.

3. Determine which of the following are vector spaces over $\mathbb{R}$.

(a) The set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$.

(b) The set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$.

(c) The set of all differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$.

4. Let V and W be vector spaces over a field F. The (*external*) *direct sum* of V and W is defined to be the cartesian product $V \times W$, along with an addition given by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \qquad \text{for all } v_1, v_2 \in V \text{ and } w_1, w_2 \in W,$$

and scalar multiplication given by

$$a(v, w) = (av, aw) \qquad \text{for all } v \in V,\ w \in W \text{ and } a \in F.$$

Show that the direct sum of V and W is again a vector space. (Having proved this, the direct sum of V and W is usually denoted by $V \oplus W$.)

5. Let V be a vector space over a field F. Show that if $v \in V$ and $a \in F$ satisfy $av = 0_V$, then $v = 0_V$ or $a = 0_F$.

6. Let V be a vector space over a field F and let U_1 and U_2 be subspaces of V.

(a) Prove that the intersection $U_1 \cap U_2$ is also a subspace of V.

(b) Prove that the sum

$$U_1 + U_2 = \{v_1 + v_2 \in V \mid v_1 \in U_1 \text{ and } v_2 \in U_2\}$$

is again a subspace of V.

7. Let V be a vector space over a field F and let $v_1, \ldots, v_r$ be a collection of vectors in V. Show that $\operatorname{Span}\{v_1, \ldots, v_r\}$ is the smallest subspace of V containing the vectors $v_1, \ldots, v_r$. That is, show that if U is a subspace of V such that $v_1, \ldots, v_r \in U$, then $\operatorname{Span}\{v_1, \ldots, v_r\} \subseteq U$.

8. Determine whether or not the vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

span $\mathbb{R}^3$.

9. Determine whether or not the vectors $x^2 + x + 1, x^2 + x - 1, x + 2$ span $\mathcal{P}_2(\mathbb{R})$, the set of all polynomials of degree at most 2 with coefficients in $\mathbb{R}$.

10. Let V be a vector space over a field F and let $v_1, \ldots, v_r$ be a collection of vectors that span V. Show that if each v_i can be written as a linear combination of the vectors $w_1, \ldots, w_s$, then $w_1, \ldots, w_s$ also span V.