ASSIGNMENT 3, DUE MONDAY 2 OCTOBER

1. Let V be a vector space. If U and W are subspaces of V such that

$$U + W = V \qquad \text{and} \qquad U \cap W = \{0_V\}$$

then we say that V is the *internal direct sum of U and W*. In this case we write $V = U \oplus W$. (This is the same notation as used for the external direct sum, but it means something different in this context.) Show that V is the internal direct sum of U and W if and only if every vector in V may be written uniquely in the form $u + w$ with $u \in U$ and $w \in W$.

2. Prove the following statement or give a counterexample. *If U_1, U_2, W are subspaces of V such that $V = U_1 \oplus W$ and $V = U_2 \oplus W$, then $U_1 = U_2$.*

3. Determine whether or not the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

in $\mathbb{R}^3$ are linearly independent.

4. Determine whether or not the vectors $x^2 + 2x + 1, 3x^2 + x + 7, x^2 - 3x + 5$ in $\mathcal{P}_2(\mathbb{R})$ are linearly independent.

5. Let F be a field. Is it always true that the vectors

$$\begin{pmatrix} 1_F \\ 1_F \end{pmatrix}, \begin{pmatrix} 1_F \\ -1_F \end{pmatrix}$$

in F^2 are linearly independent? Explain your reasoning.

6. Prove that if a collection of vectors $v_1, \ldots, v_r$ in V is linearly dependent and v is any vector in V, then the collection $v_1, \ldots, v_r, v$ is also linearly dependent.

7. If $v_1, \ldots, v_r$ is a linearly independent collection of vectors, is it always true that the collection $v_1, v_1 + v_2, v_1 + v_3, \ldots, v_1 + v_r$ is linearly independent? Justify your answer.

8. Recall that a *plane* in $\mathbb{R}^3$ is the set of all points $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ satisfying a fixed equation

$$ax_1 + bx_2 + cx_3 = d$$

where a, b, c, d are real numbers and at least one of a, b, c is non-zero. Now consider the pair of vectors

$$\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \in \mathbb{R}^3.$$

Find an equation of the plane that contains the origin and in which the above two vectors lie. Then use your equation to show that the span of these vectors is in fact equal to that plane. (Hint: To prove that two sets are equal, you need to show that the first is contained in the second, and vice versa.)