ASSIGNMENT 4, DUE MONDAY 9 OCTOBER

1. Consider the collection

$$S = \{x^2 + x + 1,\ 2x^2 + x,\ x + 2,\ x^2 + 2x + 3\}$$

of polynomials with rational coefficients. Determine whether or not S is linearly independent. If not, then express one of the vectors in S as a linear combination of the others and remove that vector from the collection. Repeat this until you obtain a minimal spanning set for $\operatorname{Span} S$.

2. Mimic the procedure in the replacement theorem to find a basis of $\mathbb{Q}^4$ that contains the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ 2 \end{pmatrix}.$$

3. Recall that $\mathcal{P}_3(F)$ is the set of all polynomials over a field F of degree at most 3. Prove or disprove the following statement. *There exists a basis f_0, f_1, f_2, f_3 of $\mathcal{P}_3(F)$ such that none of the f_i has degree 2.*

4. Prove or disprove the following statement. *If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3, v_4 \notin U$, then v_1, v_2 is a basis of U.*

5. Prove that if $f_0, \ldots, f_n$ is a collection of vectors in $\mathcal{P}_n(\mathbb{R})$ such that $f_i(e^\pi) = 0$ for all $0 \leq i \leq n$, then $f_0, \ldots, f_n$ cannot be linearly independent.

6. Let V and W be finite dimensional vector spaces with chosen bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$, respectively. Prove that $\dim(V \oplus W) = n + m$ by showing that

$$(v_1, 0_W),\ \ldots,\ (v_n, 0_W),\ (0_V, w_1),\ \ldots,\ (0_V, w_m)$$

is a basis of the external direct sum $V \oplus W$.