ASSIGNMENT 5, DUE MONDAY 16 OCTOBER

1. Prove that the set $\mathbb{C}$ of complex numbers is a vector space over $\mathbb{R}$. What is the dimension of $\mathbb{C}$ as an $\mathbb{R}$-vector space? Justify your answer.

2. Let V be an n-dimensional vector space and let U be a subspace of V such that $\dim U = n$. Show that $U = V$. (This seems obvious, but it's not!)

3. Let U and W be subspaces of a vector space V. Prove that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

(Hint: Start by choosing a basis of $U \cap W$, then use the replacement theorem to extend it to bases of U and W, respectively.)

4. Use the result in the previous question to prove that any two distinct planes in $\mathbb{R}^3$ that pass through the origin must intersect in a line.

5. Let θ be a real number and let $\rho_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates the x-y-plane counter-clockwise about the origin by θ radians.

(a) For a vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, use trigonometric identities to describe $\rho_\theta\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$ in terms of only x, y and θ. (Do not do this in terms of matrices. We haven't gotten there yet.)

(b) Use part (a) to prove that ρ_θ is a linear transformation.

6. Prove that the function $\sigma \colon \mathbb{R}^2 \to \mathbb{R}^2$ that reflects the x-y-plane about the line $y = 2x$ is a linear transformation.

7. Show that if $S \colon U \to V$ and $T \colon V \to W$ are linear transformations, then the composition $T \circ S \colon U \to W$ is also a linear transformation.

8. Suppose that a vector space V is the internal direct sum of two of its subspaces U and W. Recall that any vector in V can be written in the form $u + w$ with $u \in U$ and $w \in W$. Prove that the function $\pi \colon V \to U$ given by $\pi(u + w) = u$ is a linear transformation. (Hint: Be sure to verify that π never assigns two different outputs to the same input.) We call π the *projection of V onto U*.

9. Let V and W be vector spaces over a field F and let $T\colon V \to W$ be a linear transformation. Prove that if U is a subspace of W, then the subset

$$T^{-1}(U) = \{v \in V \mid T(v) \in U\}$$

of V is in fact a subspace of V. Use this to deduce that the kernel of T is a subspace of V.

10. Let V and W be vector spaces over a field F and let $T\colon V \to W$ be a linear transformation. Prove that if U is a subspace of V, then the subset

$$T(U) = \{w \in W \mid \text{there exists } v \in U \text{ such that } w = T(v)\}$$

of W is in fact a subspace of W. Use this to deduce that the image of T is a subspace of W.