ASSIGNMENT 6, DUE WEDNESDAY 1 NOVEMBER

1. Determine if/when the following linear transformations are injective or surjective.

(a) The identity transformation $\mathrm{id}_V : V \to V$.

(b) The zero transformation $Z : V \to W$ given by $Z(v) = 0_W$.

(c) The projection onto the x-y-plane $\pi : \mathbb{R}^3 \to \mathbb{R}^3$.

(d) The formal derivative $\frac{d}{dx} : \mathcal{P}_n(\mathbb{Q}) \to \mathcal{P}_{n-1}(\mathbb{Q})$.

(e) The formal integral $\int : \mathcal{P}_{n-1}(\mathbb{Q}) \to \mathcal{P}_n(\mathbb{Q})$.

2. Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Show that if the composition $g \circ f : X \to Z$ is injective, then f is injective.

(b) Show that if $g \circ f$ is surjective, then g is surjective.

3. Let V be an n-dimensional vector space and W an m-dimensional vector space.

(a) Use the rank-nullity theorem to prove that if $m > n$, then there cannot exist a surjective linear transformation from V to W.

(b) Use the rank-nullity theorem to prove that if $m < n$, then there cannot exist an injective linear transformation from V to W.

4. Compute the matrix that represents the formal derivative

$$\frac{d}{dx} : \mathcal{P}_3(\mathbb{Q}) \longrightarrow \mathcal{P}_2(\mathbb{Q})$$

with respect to the bases

$$x + 1,\ x^2 + x + 1,\ x^2 + x,\ x^3 + x \qquad \text{of } \mathcal{P}_3(\mathbb{Q})$$

and

$$x^2 - 1,\ x^2 - x,\ x + 1 \qquad \text{of } \mathcal{P}_2(\mathbb{Q}).$$

(Caution: The order of these bases will affect the answer.)

5. Let $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ be the counter-clockwise rotation of the x-y-plane about the origin by $\frac{2\pi}{3}$ radians and let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = 2x$.

(a) Compute the matrices that represent ρ and σ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2$ of $\mathbb{R}^2$ (viewed as the basis of both the domain and codomain).

(b) Use part (a) to compute the matrices representing $\sigma \circ \rho$ and $\rho \circ \sigma$ with respect to $\mathbf{e}_1, \mathbf{e}_2$.

6. Let V be an n-dimensional vector space and let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V. Compute the matrix that represents the identity transformation $\mathrm{id}_V : V \to V$ with respect to $\mathcal{B}$, viewed as the basis of both the domain and codomain.

7. Let V and W be vector spaces over a field F and let $\mathcal{L}(V, W)$ denote the set of all linear transformations from V to W. For $S, T \in \mathcal{L}(V, W)$, define the *sum* of S and T to be the linear transformation $S + T : V \to W$ given by

$$(S + T)(v) = S(v) + T(v) \qquad \text{for all } v \in V.$$

For $T \in \mathcal{L}(V, W)$ and $a \in F$, define the *scalar multiple* of T by a to be the linear transformation $aT : V \to W$ given by

$$(aT)(v) = a\,T(v) \qquad \text{for all } v \in V.$$

Prove that these operations give $\mathcal{L}(V, W)$ the structure of a vector space over F.

8. Let A and B be sets and $f : A \to B$ a function. Prove that f is a bijection if and only if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In this case we call g an *inverse function* for f.

9. Let $T : V \to W$ be a bijective linear transformation and let $S : W \to V$ be an inverse function for T. (See the previous exercise.) Prove that S is also a linear transformation.

10. Let $n \geq 1$ and consider the formal derivative and integral linear transformations

$$\tfrac{d}{dx} : \mathcal{P}_n(\mathbb{Q}) \to \mathcal{P}_{n-1}(\mathbb{Q}), \qquad \int : \mathcal{P}_{n-1}(\mathbb{Q}) \to \mathcal{P}_n(\mathbb{Q}).$$

Without computing anything, determine whether or not they are mutually inverse. Are either of their compositions equal to the identity transformation?