ASSIGNMENT 7, DUE FRIDAY 10 NOVEMBER

1. Show that if a matrix has a left inverse and a right inverse, then the two respective inverses must be equal. In other words, show that if A, C and D are $n \times n$ matrices such that $CA = I$ and $AD = I$, then $C = D$.

2. Use the invertible matrix theorem to prove that if A and B are $n \times n$ matrices such that AB is invertible, then A and B are both invertible.

3. An $n \times n$ matrix A is *non-singular* if the equation $A\mathbf{x} = \mathbf{0}$ has only the zero solution. By the invertible matrix theorem, this is equivalent to A being invertible, but **do not** use that characterisation for this exercise.

(a) Let A and B be $n \times n$ matrices such that A is non-singular and $AB = 0$, the zero matrix. Show that $B = 0$. (Hint: Think about the columns of B.)

(b) Use the result of part (a) to prove that if A, B and C are $n \times n$ matrices such that A is non-singular and $AB = AC$, then $B = C$.

4. Compute A^{-1}, where

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 3 & -2 \\ -1 & 1 & 4 \end{pmatrix}.$$

5. Let X be a set. A *relation* on X is a subset $R \subseteq X \times X$. A relation R on X is an *equivalence relation* if the following properties hold.

(i) For all $x \in X$, $(x, x) \in R$.

(ii) For all $x, y \in X$, if $(x, y) \in R$ then $(y, x) \in R$.

(iii) For all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

(Property (i) says that R is *reflexive*, (ii) says that R is *symmetric* and (iii) says that R is *transitive*.)

Now let $\mathrm{Mat}_n(F)$ denote the set of all $n \times n$ matrices with entries in F. Prove that the relation

$$R = \{(A, B) \in \mathrm{Mat}_n(F) \times \mathrm{Mat}_n(F) \mid B \text{ is similar to } A\}$$

is an equivalence relation on $\mathrm{Mat}_n(F)$.

6. Let A and B be $n \times n$ matrices and $i \geq 0$ an integer. Show that if B is similar to A, then B^i is similar to A^i. (By convention we set $A^0 = I$.)

7. Let $T\colon \mathcal{P}_4(\mathbb{Q}) \to \mathcal{P}_2(\mathbb{Q})$ be the linear transformation represented by the matrix

$$\begin{pmatrix} 2 & -2 & 3 & 1 & 1 \\ -1 & 1 & 3 & 4 & -5 \\ 2 & -2 & 0 & -2 & 4 \end{pmatrix}$$

with respect to the bases

$$x^4,\ x^4 + x^3,\ x^4 + x^2,\ x^4 + x,\ x^4 + 1 \qquad \text{and} \qquad x^2 + x,\ x + 1,\ 1$$

of $\mathcal{P}_4(\mathbb{Q})$ and $\mathcal{P}_2(\mathbb{Q})$, respectively. Find bases of $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$.

8. (Super challenge problem.) Find a 4×4 matrix whose null space has basis

$$\begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}$$

and whose column space has basis

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$