ASSIGNMENT 8, DUE FRIDAY 17 NOVEMBER

1. Construct a linear transformation and use it to find a basis of the subspace

$$\operatorname{Span}\left\{x^2 - x + 2,\ -3x^2 + 3x - 6,\ x + 2,\ -2x^2 + 3x - 2,\ x^2 - 5x - 6\right\}$$

of $\mathcal{P}_2(\mathbb{Q})$.

2. Compute the determinant of the matrix

$$\begin{pmatrix} 1 & -1 & 3 & -1 \\ -2 & 1 & 0 & 1 \\ 3 & 0 & -2 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}.$$

3. Compute the determinant of the matrix

$$A = \begin{pmatrix} \overline{4} & \overline{2} & \overline{4} \\ \overline{0} & \overline{5} & \overline{1} \\ \overline{6} & \overline{0} & \overline{2} \end{pmatrix}$$

with coefficients in the field $\mathbb{Z}/7$. Is A invertible? If so, then compute A^{-1}.

4. Use the definition of determinant in terms of cofactor expansion to prove that if A is an $n \times n$ matrix and B is a matrix obtained from A by adding a scalar multiple of one row (or column) to another, then $\det B = \det A$.

5. For an invertible $n \times n$ matrix A, prove that $\det A^{-1} = \frac{1}{\det A}$.

6. Show that if A and B are similar $n \times n$ matrices, then $\det A = \det B$.

7. An $n \times n$ matrix $A = (a_{ij})$ is said to be *upper triangular* if $a_{ij} = 0$ whenever $i > j$. If A is upper triangular, show that $\det A = \prod_{i=1}^{n} a_{ii}$.

8. Find all eigenvalues of the matrix

$$\begin{pmatrix} -1 & 6 & 2 \\ 0 & 5 & -6 \\ 1 & 0 & -2 \end{pmatrix}.$$

For each eigenvalue, record its algebraic multiplicity and calculate its eigenspace. What are the respective geometric multiplicities of these eigenvalues?

9. Show that if A is an $n \times n$ matrix and B is similar to A, then $\chi_B(x) = \chi_A(x)$. Use this to deduce that similar matrices have the same eigenvalues.

10. Let V be an n-dimensional vector space and $T\colon V \to V$ a linear transformation. Choosing a basis $\mathcal{B}$ of V, we propose the following definitions.

- The *determinant* of T is the determinant of the $n \times n$ matrix $M_{\mathcal{B}}^{\mathcal{B}}(T)$.
- The *characteristic polynomial* of T is the characteristic polynomial of $M_{\mathcal{B}}^{\mathcal{B}}(T)$.
- An *eigenvalue* of T is an eigenvalue of $M_{\mathcal{B}}^{\mathcal{B}}(T)$.

Show that each of the above definitions is independent of the choice of $\mathcal{B}$. In other words, show that if $\mathcal{C}$ is another basis of V, then the above concepts for T, defined in terms of $\mathcal{C}$, are the same as those defined in terms of $\mathcal{B}$. (Hint: Think change of basis.)