ASSIGNMENT 9, DUE FRIDAY 1 DECEMBER

1. Let A be an $n \times n$ matrix over a field F. Prove that the constant term in the characteristic polynomial of A is the determinant of A.

2. Show that the matrix

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 0 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix}$$

is diagonalisable by finding a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

3. Without making a computation, find the algebraic and geometric multiplicities of the eigenvalues of the Jordan canonical form matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Explain your reasoning.

4. Prove or disprove the following: *If two $n \times n$ matrices have the same eigenvalues with the same respective algebraic and geometric multiplicities, then they must be similar.* (Hint: Think about Jordan canonical forms.)

5. Compute the Jordan canonical form J of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

Also, find an invertible matrix P such that $A = PJP^{-1}$.

6. Find the Jordan canonical form J of the matrix

$$A = \begin{pmatrix} 4 & -3 & 3 & -2 \\ -1 & 0 & -1 & 1 \\ -4 & 1 & -2 & 4 \\ 2 & -3 & 3 & 0 \end{pmatrix}.$$

Also, find an invertible matrix P such that $A = PJP^{-1}$.

7. Find the Jordan canonical form J of the matrix

$$A = \begin{pmatrix} 5 & 0 & 1 & 1 \\ -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 \\ -4 & -1 & -2 & 2 \end{pmatrix}.$$

Also, find an invertible matrix P such that $A = PJP^{-1}$.

8. Consider a 3×3 Jordan block matrix

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

One can show using induction and infinite series that

$$\exp(Jt) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

Now let A be the 3×3 matrix given in Exercise 5.

(a) Use Jordan canonical forms to compute $\exp(A)$.

(b) Use Jordan canonical forms to find the general solution to the system of first order linear differential equations

$$\mathbf{x}' = A\mathbf{x}.$$

Then find the particular solution that satisfies the initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$