NOTES ON SPACES AND BASES

Throughout these notes we will be discussing certain finite collections of vectors in vector spaces. If the vectors in such a collection are $v_1, \dots, v_r$, then we may denote the collection simply by $v_1, \dots, v_r$. However, it will be convenient at times to consider a collection as an actual set and to use set theoretic language when talking about it. In these cases we shall denote such a collection by $\{v_1, \dots, v_r\}$. Both notations mean the same thing.

Definition 1. Let V be a vector space and let $v_1, \dots, v_r$ be a collection of vectors in V. We define the *span* of $v_1, \dots, v_r$ to be the set

$$\operatorname{Span}\{v_1, \dots, v_r\} = \{a_1 v_1 + \cdots + a_r v_r \in V \mid a_1, \dots, a_r \in F\}.$$

In other words, $\operatorname{Span}\{v_1, \dots, v_r\}$ is the set of all linear combinations of $v_1, \dots, v_r$.

Proposition 2. *If V is a vector space and $v_1, \dots, v_r$ is a collection of vectors in V, then $\operatorname{Span}\{v_1, \dots, v_r\}$ is a subspace of V.*

Definition 3. Let V be a vector space and $v_1, \dots, v_r$ a collection of vectors in V. We say that $v_1, \dots, v_r$ *span* V if $\operatorname{Span}\{v_1, \dots, v_r\} = V$. In other words, $v_1, \dots, v_r$ span V if and only if every element of V can be written as a linear combination of the vectors $v_1, \dots, v_r$. In this case, we call $v_1, \dots, v_r$ a *spanning set* for V.

Definition 4. A vector space V is *finite dimensional* if there exists a finite collection of vectors $v_1, \dots, v_r \in V$ that span V, that is, if V has a finite spanning set.

Definition 5. Let V be a vector space and $v_1, \dots, v_r$ a collection of vectors in V. We say that $v_1, \dots, v_r$ are *linearly independent* if the only way to write the zero vector as a linear combination

$$a_1 v_1 + \cdots + a_r v_r = 0_V$$

is with all of the coefficients $a_1 = \cdots = a_r = 0_F$.

Definition 6. Let V be a finite dimensional vector space. A collection of vectors $v_1, \dots, v_r$ in V is a *basis* of V if it is linearly independent and spans V.

Definition 7. A collection of vectors $v_1, \dots, v_r$ in a vector space V is a *minimal spanning set* for V if $v_1, \dots, v_r$ spans V, and if no proper subset of $\{v_1, \dots, v_r\}$ spans V. (A proper subset of a set $\mathcal{A}$ is a subset $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B} \neq \mathcal{A}$.)

Proposition 8. *If V is a vector space and $v_1, \dots, v_r$ is a minimal spanning set for V, then $v_1, \dots, v_r$ is a basis of V.*

Proof. Suppose, to the contrary of what we want to prove, that $v_1, \ldots, v_r$ is *not* a basis of V. Then because $v_1, \ldots, v_r$ span V, they must be linearly dependent. This means that we can find coefficients $a_1, \ldots, a_r \in F$, not all zero, such that

$$a_1 v_1 + \cdots + a_r v_r = 0_V.$$

Choosing one of the non-zero coefficients, say a_i, we can solve for v_i and obtain

$$v_i = -\frac{a_1}{a_i} v_1 - \cdots - \frac{a_{i-1}}{a_i} v_{i-1} - \frac{a_{i+1}}{a_i} v_{i+1} - \cdots - \frac{a_r}{a_i} v_r,$$

where v_i does not appear on the right hand side. Now let v be any vector in V. Because $v_1, \ldots, v_r$ span V, we can write v as a linear combination

$$v = b_1 v_1 + \cdots + b_r v_r.$$

Substituting the above expression for v_i, this becomes

$$v = b_1 v_1 + \cdots + b_i \left(-\frac{a_1}{a_i} v_1 - \cdots - \frac{a_{i-1}}{a_i} v_{i-1} - \frac{a_{i+1}}{a_i} v_{i+1} - \cdots - \frac{a_r}{a_i} v_r \right) + \cdots + b_r v_r.$$

The right hand side is a linear combination of $v_1, \ldots, v_r$ that does not involve v_i. In other words, v is a linear combination of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r$. Since v was arbitrarily chosen, this means that the vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r$ span V. But then

$$\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r\}$$

is a proper subset of $\{v_1, \ldots, v_r\}$ that spans V. This contradicts the fact that $v_1, \ldots, v_r$ is a minimal spanning set, so the statement that $v_1, \ldots, v_r$ is not a basis must be false. □

Corollary 9. *Every finite dimensional vector space V has a basis.*

Proof. Let $\mathcal{S} = \{v_1, \ldots, v_r\}$ be a finite spanning set for V. We claim that $\mathcal{S}$ contains a basis of V. If $\mathcal{S}$ is a minimal spanning set, then by the previous proposition, it is already a basis. If it is not minimal, then there exists some proper subset $\mathcal{S}_1 \subsetneq \mathcal{S}$ that also spans V. If $\mathcal{S}_1$ is a minimal spanning set, then it is a basis. If $\mathcal{S}_1$ is not minimal, then it contains a proper subset $\mathcal{S}_2 \subsetneq \mathcal{S}_1$ that also spans V. Continuing this way, we will eventually obtain a subset $\mathcal{S}_n$ of $\mathcal{S}$ that is minimal (hence a basis), or we will construct an infinite nested sequence of proper subsets

$$\mathcal{S} \supsetneq \mathcal{S}_1 \supsetneq \mathcal{S}_2 \supsetneq \cdots,$$

each of which spans V. But the second scenario cannot take place, because $\mathcal{S}$ is a finite set, thus has only finitely many subsets. □

The following result illustrates why bases are so important.

Proposition 10. *Let V be a finite dimensional vector space and let $v_1, \ldots, v_n$ be a basis of V. Then any vector in V can be expressed uniquely as a linear combination of $v_1, \ldots, v_n$.*

Proof. If v is any vector in V, then because $v_1, \dots, v_n$ span V, we may write v as a linear combination

$$v = a_1v_1 + \cdots + a_nv_n.$$

The uniqueness statement in the proposition has to do with the fact that the coefficients a_i are unique. To see this, choose another expression of v as a linear combination

$$v = b_1v_1 + \cdots + b_nv_n.$$

We then have

$$0_V = v - v = a_1v_1 + \cdots + a_nv_n - (b_1v_1 + \cdots + b_nv_n) = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n.$$

Since $v_1, \dots, v_n$ are linearly independent, we must have $a_i - b_i = 0_F$ for all i. In other words, $a_i = b_i$ for all i, hence there is really only one way to express v as a linear combination of $v_1, \dots, v_n$. □

The following theorem is what makes everything 'go' in the theory of vector spaces. Its importance cannot be overstated, which is why I have just stated again how important it is.

Theorem 11 (The replacement theorem)**.** *Let V be a finite dimensional vector space. If*

$$u_1, \dots, u_m$$

is a collection of linearly independent vectors in V, then there exists a basis of V containing $u_1, \dots, u_m$.

To make things look cleaner in the proof, we need to establish some notation. If $v_1, \dots, v_r$ is any collection of vectors in V, then we denote by $v_1, \dots \overset{i}{\uparrow} \dots, v_r$ the collection of vectors obtained by removing the vector v_i from $v_1, \dots, v_r$.

Proof. Because V is finite dimensional, we know from the previous corollary that there exists some basis $v_1, \dots, v_n$ of V. Because $v_1, \dots, v_n$ span V, we can write the last vector u_m as a linear combination

$$u_m = a_1v_1 + \cdots + a_nv_n. \tag{1}$$

Now u_m cannot be the zero vector, otherwise $u_1, \dots, u_m$ would not be linearly independent. This means that some coefficient a_{i_m} in the above expression is non-zero. We claim that the collection

$$u_m, v_1, \dots \overset{i_m}{\uparrow} \dots, v_n \tag{2}$$

is also a basis of V. (Here we have added u_m to the collection $v_1, \dots, v_n$, but removed the vector v_{i_m}.) To show that the collection (2) spans V, observe that since $a_{i_m} \neq 0_F$, we can solve the above equation and obtain

$$v_{i_m} = \frac{1}{a_{i_m}}u_m - \frac{a_1}{a_{i_m}}v_1 - \cdots - \frac{a_{i_m-1}}{a_{i_m}}v_{i_m-1} - \frac{a_{i_m+1}}{a_{i_m}}v_{i_m+1} \cdots - \frac{a_n}{a_{i_m}}v_n,$$

where v_{i_m} does not appear on the right hand side. This means that v_{i_m} is a linear combination of the vectors in (2). Because all other v_i are clearly linear combinations of the vectors in (2), it follows that

$$\operatorname{Span}\{u_m, v_1, \ldots \overset{i_m}{\uparrow} \ldots, v_n\} = \operatorname{Span}\{v_1, \ldots, v_n\} = V.$$

We now need to show that the collection (2) is linearly independent. Suppose that

$$bu_m + b_1v_1 + \cdots + b_{i_m-1}v_{i_m-1} + b_{i_m+1}v_{i_m+1} + \cdots + b_nv_n = 0_V,$$

where v_{i_m} does not appear in the left hand expression. Substituting (1) into this expression yields

$$b\,(a_1v_1 + \cdots + a_nv_n) + b_1v_1 + \cdots + b_{i_m-1}v_{i_m-1} + b_{i_m+1}v_{i_m+1} + \cdots + b_nv_n = 0_V.$$

The left hand side is now a linear combination of the vectors $v_1, \ldots, v_n$, and the coefficient on v_{i_m} is ba_{i_m}. Because $v_1, \ldots, v_n$ are linearly independent, we must have $ba_{i_m} = 0_F$, and since $a_{i_m} \neq 0_F$, this forces $b = 0_F$. But then

$$b_1v_1 + \cdots + b_nv_n = 0_V,$$

hence again by the linear independence of $v_1, \ldots, v_n$, all of the b_i must be zero. Combining these facts, we have shown that $b = b_1 = \cdots = b_n = 0_F$. This proves that the collection (2) is linearly independent, completing the proof that it is a basis of V.

We have now completed the first step of the replacement procedure. The next step would be to repeat the above proof, but with u_{m-1} playing the role of u_m, and $u_m, v_1, \ldots \overset{i_m}{\uparrow} \ldots, v_n$ playing the role of the basis $v_1, \ldots, v_n$. The only thing we need to make sure of is that the vector we remove from $u_m, v_1, \ldots \overset{i_m}{\uparrow} \ldots, v_n$ is one of the v_i, and not u_m. So let's revisit how this new replacement procedure would begin.

The first thing we would do is to observe that since $u_m, v_1, \ldots \overset{i_m}{\uparrow} \ldots, v_n$ is a basis of V, we can write u_{m-1} as a linear combination of $u_m, v_1, \ldots \overset{i_m}{\uparrow} \ldots, v_n$. The key fact is that, in this linear combination, the coefficient on some v_i must be non-zero. Otherwise, u_{m-1} would be a linear combination (scalar multiple) of the vector u_m, and this would contradict the fact that $u_1, \ldots, u_m$ are linearly independent. If $v_{i_{m-1}}$ is the vector having a non-zero coefficient in the linear combination, then the proof of the first replacement step reveals that

$$u_{m-1}, u_m, v_1, \ldots \overset{i_{m-1}}{\uparrow} \quad \overset{i_m}{\uparrow} \ldots, v_n$$

is also a basis of V. This establishes the second replacement step.

We would proceed with the following replacement steps similarly, each time making sure that we can replace a remaining v_i with the next u_j to get another basis, always using the property that $u_1, \dots, u_m$ are linearly independent. After the mth step, we will have a basis

$$u_1, \dots, u_m, v_1, \dots \overset{i_1}{\uparrow} \cdots \overset{i_m}{\uparrow} \dots, v_n$$

of V, that is, unless there are more of the u_j than there are v_i, and this would happen precisely when $m > n$. We now claim that this situation never takes place.

Suppose, contrary to our claim, that $m > n$. Then after the nth step of the replacement procedure, we would end up with a basis

$$u_{m-n+1}, \dots, u_m$$

of V, and the vectors left over would be $u_1, \dots, u_{m-n}$. But since $u_{m-n+1}, \dots, u_m$ would be a basis of V, we could write u_1 as a linear combination of $u_{m-n+1}, \dots, u_m$, and this would contradict the fact that $u_1, \dots, u_m$ are linearly independent. It follows that $m \leq n$. □

Corollary 12. *Let V be a finite dimensional vector space. If $u_1, \dots, u_m$ are linearly independent vectors in V and $v_1, \dots, v_n$ is a basis of V, then $m \leq n$.*

Corollary 13. *Let V be a finite dimensional vector space. If $u_1, \dots, u_m$ and $v_1, \dots, v_n$ are bases of V, then $m = n$.*

Proof. Since $u_1, \dots, u_m$ are linearly independent (they are a basis) and $v_1, \dots, v_n$ is a basis of V, Corollary 12 tells us that $m \leq n$. On the other hand, because $v_1, \dots, v_n$ are linearly independent and $u_1, \dots, u_m$ is a basis of V, the same result tells us that $n \leq m$. □

Definition 14. If V is a finite dimensional vector space, then the number of vectors in any basis of V is called the *dimension* of V. In this case, if V has dimension n, then we also say that V is an *n-dimensional* vector space. We often denote the dimension of V by $\dim V$.

The following two 'numerical' results are the standard facts that everyone should know about bases.

Proposition 15. *Let V be an n-dimensional vector space.*

(1) *Any collection of vectors in V that contains more than n elements is linearly dependent.*
(2) *Any collection of vectors in V that contains less than n elements cannot span V.*

Proof. For (1), recall that the statement of Corollary 12 is

If $v_1, \dots, v_r$ are linearly independent then $r \leq n$.

This is equivalent to the statement

If $r > n$ then $v_1, \dots, v_r$ are linearly dependent.

(Look up the term 'contrapositive' for details.)

To prove (2), let $r < n$ and suppose that $v_1, \dots, v_r$ is a collection of vectors that span V. We wish to derive a contradiction from this. Because $v_1, \dots, v_r$ span V, we know that $\{v_1, \dots, v_r\}$ contains a minimal spanning subset by the proof of Corollary 9. This minimal spanning subet will contain at most r vectors. It will also be a basis of V. But this means that V has a basis containing less than n elements, contradicting Corollary 13. □

Proposition 16. *Let V be an n-dimensional vector space.*

(1) *Any collection of n linearly independent vectors in V is a basis of V.*
(2) *Any collection of n vectors in V that spans V is a basis of V.*

Proof. For (1), if $u_1, \dots, u_n$ are linearly independent vectors in V, then by the proof of the replacement theorem, we can choose a basis $v_1, \dots, v_n$ of V and extend $u_1, \dots, u_n$ to a basis of V, replacing some v_i with a vector u_j at each step. In this situation, at the end of the replacement procedure, we would simply end up with the collection $u_1, \dots, u_n$. Because the replacement procedure always produces a basis at each step, the collection $u_1, \dots, u_n$ must have already been a basis of V.

To prove (2), suppose that $v_1, \dots, v_n$ is a collection of vectors that span V, but that do not form a basis of V. We wish to derive a contradiction from this. Because $v_1, \dots, v_n$ span V but are not a basis, Proposition 8 implies that $v_1, \dots, v_n$ cannot be a minimal spanning set for V. This means that there is some proper subset of $\{v_1, \dots, v_n\}$ that also spans V. That spanning subset will contain less than n vectors, contradicting part (2) of Proposition 15. □

Theorem 17. *If V is an n-dimensional vector space and $U \subseteq V$ is a subspace of V, then U is finite dimensional and* $\dim U \leq n$.

Proof. If U is the zero subspace, then U is clearly finite dimensional, being spanned by the zero vector. Now suppose that U is not the zero subspace. To show that U is finite dimensional, we will give a proof by contradiction. So suppose that U is not finite dimensional, i.e., that it does not have a finite spanning set. Then since U is not the zero subspace, we may choose a non-zero $v_1 \in U$. Next, we may choose a vector $v_2 \in U$ that is not a scalar multiple of v_1, for otherwise U would be spanned by v_1. Similarly, we can choose $v_3 \in U$ such that v_3 is not a linear combination of v_1, v_2, for otherwise U would be spanned by the collection v_1, v_2. Continuing in this way, we can eventually construct a collection of vectors $v_1, \dots, v_{n+1}$ in U with the property that v_{i+1} is not a linear combination of $v_1, \dots, v_i$ for all $1 \leq i \leq n$.

We claim that $v_1, \dots, v_{n+1}$ are linearly independent. To see this, suppose that

$$a_1 v_1 + \cdots + a_{n+1} v_{n+1} = 0_V. \tag{3}$$

We need to show that all of the coefficients are zero. First suppose that $a_{n+1} \neq 0_F$. Then we can solve (3) to get v_{n+1} as a linear combination of $v_1, \ldots, v_n$. This clearly cannot happen, so we must have $a_{n+1} = 0_F$. This means that (3) is of the form

$$a_1 v_1 + \cdots + a_n v_n = 0_V. \tag{4}$$

Now suppose that $a_n \neq 0_F$. Then again, we can solve (4) to get v_n as a linear combination of $v_1, \ldots, v_{n-1}$. Again, this cannot happen, so we must have $a_n = 0_F$. Continuing in this way, we eventually obtain the relations $a_2 = \cdots = a_{n+1} = 0_F$ and $a_1 v_1 = 0_V$. Now if a_1 were non-zero, then because $a_1 v_1 = 0_V$, this would force $v_1 = 0_V$, contradicting our choice of v_1. This means that we must have $a_1 = 0_F$ as well, showing that $v_1, \ldots, v_{n+1}$ are linearly independent.

We have now shown that $v_1, \ldots, v_{n+1}$ is a set of $(n+1)$ linearly independent vectors in U. But $U \subseteq V$, so this is also a set of $(n+1)$ linearly independent vectors in V. Since n is the dimension of V, this contradicts Proposition 15. It must therefore be the case that U actually does have a finite spanning set.

Having established that U is finite dimensional, it now follows from Proposition 15 that any basis of U, being a collection of linearly independent vectors in V, can contain at most n elements. In other words, the dimension of U is at most n. $\square$