ON THE GENERIC KERNEL FILTRATION FOR MODULES OF CONSTANT JORDAN TYPE

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Abstract. Let $E \cong (\mathbb{Z}/p)^2$ be an elementary abelian p-group of rank two, k an algebraically closed field of characteristic p, and let $J = J(kE)$. We investigate finitely generated kE-modules M of constant Jordan type and their generic kernels $\mathfrak{K}(M)$. In particular, we answer a question posed by Carlson, Friedlander and Suslin regarding whether or not the submodules $J^{-i}\mathfrak{K}(M)$ have constant Jordan type for all $i \geq 0$. We show that this question has an affirmative answer whenever $p = 3$ or $J^2\mathfrak{K}(M) = 0$. We also show that this question has a negative answer in general by constructing a kE-module M of constant Jordan type for $p \geq 5$ such that $J^{-1}\mathfrak{K}(M)$ does not have constant Jordan type.

1. Introduction

Let p be a prime number and let k be an algebraically closed field of characteristic p. Throughout this paper we consider an elementary abelian p-group $E = \langle g_1, \ldots, g_r \rangle \cong (\mathbb{Z}/p)^r$ of rank r, where the g_i are pairwise commuting generators of order p. We shall focus mainly on the case in which $r = 2$, although we begin our exposition in a more general context. As usual, we define generators $X_i = g_i - 1$ for the Jacobson radical $J = J(kE)$, and for a non-zero $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$ we define the element $X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r$. Because $X_\alpha^p = 0$, X_α acts on a finite dimensional kE-module M via a matrix whose Jordan canonical form contains blocks of length at most p. The *Jordan type* of X_α on M is the partition $[p]^{a_p} \ldots [1]^{a_1}$ of $\dim_k M$, where X_α acts on M via a_j Jordan blocks of length j. We say that M has *constant Jordan type* if the Jordan type of X_α on M is independent of $0 \neq \alpha \in \mathbb{A}^r(k)$.

Modules of constant Jordan type were introduced by Carlson, Friedlander and Pevtsova [6]. One of the major questions regarding such modules concerns which partitions of $\dim_k M$ are realised as the Jordan types of kE-modules of constant Jordan type. This question was investigated in the papers [1, 2, 3, 4, 5], although little progress has been made towards a full classification of realisable Jordan types.

Recently, Carlson, Friedlander and Suslin [7] have investigated the constant image property for kE-modules in any rank, and for $r = 2$ they have defined the generic kernel $\mathfrak{K}(M)$ of a kE-module M, which can be characterised as the largest submodule of M having the constant image property. One of the striking features of the generic kernel is that, whenever M has constant rank, the kernel of the action of X_α on M is contained in $\mathfrak{K}(M)$. Benson [3] used this property to prove that if a kE-module (for any r) has constant Jordan type containing no blocks of length one, then the total number of Jordan blocks is divisible by p. (This statement proves a special case of an unpublished conjecture of Rickard.)

In an attempt to obtain stronger connections between modules of constant Jordan type and their generic kernels, Carlson, Friedlander and Suslin have posed the following question.

Question 1.1 ([7, Question 8.11])**.** If a $k(\mathbb{Z}/p)^2$-module M has constant Jordan type, is it true that $J^{-i}\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$?

Here $J^{-i}\mathfrak{K}(M)$ denotes the set of elements $m \in M$ for which $J^i m \subseteq \mathfrak{K}(M)$. We should point out that the original statement of this question was given in terms of the submodules $X_1^{-i}\mathfrak{K}(M)$, but as we shall see in Proposition 2.7, these two formulations are equivalent.

In addition to the above property having been true in all prior known examples, motivation for Question 1.1 came from the observation that if M has constant Jordan type, then there exists a filtration

$$0 = J^p\mathfrak{K}(M) \subseteq J^{p-1}\mathfrak{K}(M) \subseteq \cdots \subseteq J^{-p+2}\mathfrak{K}(M) \subseteq J^{-p+1}\mathfrak{K}(M) = M$$

of M in which $J^i\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$ and for $i = -p+1$. For further details, see Theorem 8.10 of [7].

In this paper we present a thorough treatment of Question 1.1 and prove that it has an affirmative answer in the special cases where $p = 3$ or $J^2\mathfrak{K}(M) = 0$. The latter case involves a duality formula and an inductive process that allows us to use the constant image property and the generic kernel of the dual M^*. Finally, we show that Question 1.1 does not have an affirmative answer in general by constructing a $k(\mathbb{Z}/p)^2$-module such that $J^{-1}\mathfrak{K}(M)$ does not have constant Jordan type in the first interesting case, namely that in which $p \geq 5$, $J^2\mathfrak{K}(M) \neq 0$ and $J^3\mathfrak{K}(M) = 0$.

2. Constant Jordan type and the generic kernel

In this section we introduce properties of modules of constant Jordan type and the generic kernel that will be needed in the sequel. The exposition roughly follows that of [7]. We begin by fixing some notation. If M is a kE-module, we denote by M^* the k-linear dual of M. Throughout this paper we restrict our attention to the *p-restricted Lie kE*-module structure on M^* defined via the action

$$((g-1).f)(m) = -f((g-1)m) \qquad \text{for all } f \in M^*,\ g \in E,\ m \in M.$$

One should contrast this with the *group theoretic kE*-module structure given by

$$(g.f)(m) = f(g^{-1}m) \qquad \text{for all } f \in M^*,\ g \in E,\ m \in M.$$

We also consider the Heller shift $\Omega(M)$ of M, which is defined to be the kernel of the projective cover $P_M \to M$. Given these structures, we now recall some important properties enjoyed by modules of constant Jordan type.

Theorem 2.1. (i) *If M is a kE-module of constant Jordan type, then M^* and $\Omega(M)$ have constant Jordan type.*
(ii) *If $M \oplus N$ has constant Jordan type, then M and N have constant Jordan type.*
(iii) *If M and N have constant Jordan type, then $M \otimes_k N$ has constant Jordan type.*

Proof. These are Propositions 5.2 and 1.8, Theorem 3.7 and Corollary 4.3 of [6]. We remark that these results are independent of the choice of Hopf algebra structure on kE. □

Definition 2.2. If M is a kE-module, we say that M has the *constant image property* if the image of X_α on M is independent of $0 \neq \alpha \in \mathbb{A}^r(k)$.

It is clear from the definition that M has the constant image property if and only if $X_\alpha M = JM$ for all non-zero $\alpha \in \mathbb{A}^r(k)$. The following proposition illustrates the connection between the constant Jordan type and constant image properties.

Proposition 2.3. (i) *If M has the constant image property, then $X_\alpha^i M = J^i M$ for all $0 \neq \alpha \in \mathbb{A}^2(k)$ and all $i \geq 0$.*

(ii) *If M has the constant image property, then M has constant Jordan type.*

(iii) *The class of kE-modules with the constant image property is closed under taking finite direct sums, quotients and radicals.*

Proof. Parts (i) and (ii) are Proposition 2.8 of [7]. Part (iii) follows from the definition of the constant image property. □

Recall that a kE-module M has *constant rank* if the rank of X_α on M is independent of $0 \neq \alpha \in \mathbb{A}^r(k)$. In particular, if M has constant Jordan type then M has constant rank. Because the action of X_α on M^* is given by the negative transpose of the action of X_α on M for any non-zero $\alpha \in \mathbb{A}^r(k)$, it follows that M^* has constant rank whenever M does.

Throughout the rest of this paper (unless otherwise specified) we shall assume that $r = 2$, that is, $E = \langle g_1, g_2 \rangle \cong (\mathbb{Z}/p)^2$ and the Jacobson radical $J = J(kE)$ is generated by the elements $X_1 = g_1 - 1$ and $X_2 = g_2 - 1$.

Definition 2.4. Let M be a finitely generated kE-module. For any $0 \neq \alpha = (\lambda_1, \lambda_2) \in \mathbb{A}^2(k)$, note that $\mathsf{Ker}(X_\alpha, M)$ is uniquely determined by the class $\bar{\alpha} \in \mathbb{P}^1(k)$. For a subset $S \subseteq \mathbb{P}^1(k)$, we denote by ${}_S M$ the submodule $\sum_{\bar{\alpha} \in S} \mathsf{Ker}(X_\alpha, M)$ of M. We define the *generic kernel* of M to be the submodule

$$\mathfrak{K}(M) = \bigcap_{S \subseteq \mathbb{P}^1(k) \text{ cofinite}} {}_S M.$$

Remark 2.5. A subset $S \subseteq \mathbb{P}^1(k)$ is *cofinite* if its complement in $\mathbb{P}^1(k)$ is a finite set. Since M is finite dimensional, there always exists a cofinite $S \subseteq \mathbb{P}^1(k)$ for which $\mathfrak{K}(M) = {}_S M$.

The following theorem shows that the generic kernel $\mathfrak{K}(M)$ is particularly well behaved in the case where M has constant rank.

Theorem 2.6. (i) *If M is a kE-module, then $\mathfrak{K}(M)$ has the constant image property. Moreover, if N is any submodule of M having the constant image property, then N is contained in $\mathfrak{K}(M)$.*

(ii) *M has constant rank if and only if $\mathfrak{K}(M) = {}_{\mathbb{P}^1(k)} M$. In this case, $\mathsf{Ker}(X_\alpha, M)$ is contained in $\mathfrak{K}(M)$ for each $0 \neq \alpha \in \mathbb{A}^2(k)$.*

Proof. This follows from Theorem 7.10 and Corollary 7.7 of [7], along with the definition of the generic kernel. □

We now use Theorem 2.6 to prove a proposition that allows us to replace 'J' by 'X_1' in the statement of Question 1.1; hence our formulation is equivalent to that given in [7]. The reader should compare this with Proposition 2.3 (i).

Proposition 2.7. *If M is a kE-module of constant rank, then for all $0 \neq \alpha \in \mathbb{A}^2(k)$ and all $i \geq 0$ we have*

$$X_\alpha^{-i} \mathfrak{K}(M) = J^{-i} \mathfrak{K}(M).$$

Proof. We proceed by induction on i, beginning with the case $i = 1$. For this it suffices to show that if $m \in M$ and $X_\alpha m \in \mathfrak{K}(M)$, then $Jm \subseteq \mathfrak{K}(M)$. Note for any $\beta \in \mathbb{A}^2(k)$ that

$X_\beta X_\alpha m \in J\mathfrak{K}(M)$, hence by Theorem 2.6 (i) and the constant image property there exists $m' \in \mathfrak{K}(M)$ such that $X_\alpha m' = X_\beta X_\alpha m = X_\alpha X_\beta m$. Thus

$$m' - X_\beta m \in \mathsf{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$$

by Theorem 2.6 (ii). It follows that $X_\beta m$ also lies in $\mathfrak{K}(M)$, showing that $Jm \subseteq \mathfrak{K}(M)$.

Now let $i > 1$ and assume by induction that $X_\alpha^{-i}\mathfrak{K}(M) = J^{-i}\mathfrak{K}(M)$. We clearly have $J^{-i-1}\mathfrak{K}(M) \subseteq X_\alpha^{-i-1}\mathfrak{K}(M)$, so suppose $m \in X_\alpha^{-i-1}\mathfrak{K}(M)$. Then $X_\alpha^i X_\alpha m = X_\alpha^{i+1} m \in \mathfrak{K}(M)$ so that $X_\alpha m \in X_\alpha^{-i}\mathfrak{K}(M) = J^{-i}\mathfrak{K}(M)$. Hence $X_\alpha J^i m = J^i X_\alpha m \subseteq \mathfrak{K}(M)$, that is, $J^i m \subseteq X_\alpha^{-1}\mathfrak{K}(M) = J^{-1}\mathfrak{K}(M)$ by the first step of the proof. It follows that $J^{i+1}m = JJ^i m \subseteq \mathfrak{K}(M)$ as required. □

The following corollary will be used implicitly in Section 4 whenever we invoke the duality provided by Theorem 3.3.

Corollary 2.8. *If M is a kE-module of constant rank, then*

$$M = J^{-p+1}\mathfrak{K}(M)/J^p\mathfrak{K}(M).$$

Proof. For all $0 \neq \alpha \in \mathbb{A}^2(k)$ we have $X_\alpha^p M = 0$, hence

$$X_\alpha^{p-1} M \subseteq \mathsf{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$$

by Theorem 2.6 (ii). It follows that $M \subseteq X_\alpha^{-p+1}\mathfrak{K}(M) = J^{-p+1}\mathfrak{K}(M)$ by Proposition 2.7. Finally, we have $J^p\mathfrak{K}(M) = 0$ by part (i) of Proposition 2.3. □

3. Duality for generic kernels

In this section we introduce a duality formula that will be used to study Question 1.1 in terms of the generic kernels of both M and M^*. This is a slightly different approach to duality than that given in [7], although the underlying machinery is essentially the same. We begin with a couple of lemmas. Recall that if M is a kE-module and N is a submodule of M, then the orthogonal space

$$N^\perp = \{f \in M^* \mid f(n) = 0 \text{ for all } n \in N\}$$

is naturally a kE-submodule of M^*, and we have $(N^\perp)^\perp \cong N$.

Lemma 3.1. *If M is a kE-module of constant rank, then*

$$\mathfrak{K}(M^*) \cong (M/J\mathfrak{K}(M))^*.$$

Proof. By Proposition 8.4 and Theorem 8.6 of [7] we have

$$\mathfrak{K}(M^*) = (X_1\mathfrak{K}(M))^\perp = (J\mathfrak{K}(M))^\perp \cong (M/J\mathfrak{K}(M))^*$$

since $\mathfrak{K}(M)$ has the constant image property. □

Lemma 3.2. *If M is a kE-module in any rank r and N is a submodule of M, then*

(i) $J^{-1}(N^\perp) = (JN)^\perp$ *and*
(ii) $J(N^\perp) = (J^{-1}N)^\perp$.

Proof. First note for any $1 \neq g \in E$ and any $f \in M^*$ that the element $X = g - 1 \in kE$ acts on f via the relation

$$(X.f)(m) = -f(Xm) \qquad \text{for all } m \in M.$$

Similarly, we have $f(Xm) = -(X.f)(m)$ for all $m \in M$.

To prove (i), suppose $f \in J^{-1}(N^\perp)$ and $n \in JN$. Writing

$$n = X_1 n_1 + \cdots + X_r n_r$$

with $n_i \in N$, we have

$$f(n) = f(X_1 n_1) + \cdots + f(X_r n_r) = -(X_1.f)(n_1) - \cdots - (X_r.f)(n_r) = 0$$

since $X_i.f \in N^\perp$, hence $f \in (JN)^\perp$. Conversely, if $f \in (JN)^\perp$ and $n \in N$, then

$$(X_i.f)(n) = -f(X_i n) = 0 \qquad \text{for all } 1 \leq i \leq r$$

since $X_i n \in JN$. It follows that $X_i.f \in N^\perp$ for all i, that is, $J.f \subseteq N^\perp$.

To prove (ii), note that

$$J^{-1}(N) = J^{-1}((N^\perp)^\perp) = (J(N^\perp))^\perp$$

by (i) so that $(J^{-1}N)^\perp = J(N^\perp)$. □

We are now in a position to prove the main theorem of this section.

Theorem 3.3. *If M is a kE-module of constant rank, then for all $a, b \in \mathbb{Z}$ with $a \leq b$ we have*

$$J^a\mathfrak{K}(M^*)/J^b\mathfrak{K}(M^*) \cong (J^{-b+1}\mathfrak{K}(M)/J^{-a+1}\mathfrak{K}(M))^*.$$

Proof. We fist claim for all $a > 0$ that $J^{-a}J\mathfrak{K}(M) = J^{-a+1}\mathfrak{K}(M)$. If $m \in J^{-a+1}\mathfrak{K}(M)$, then $J^{a-1}m \subseteq \mathfrak{K}(M)$ so that $J^a m \subseteq J\mathfrak{K}(M)$ and $m \in J^{-a}J\mathfrak{K}(M)$. Conversely, suppose $J^a m \subseteq J\mathfrak{K}(M)$ and let $Y \in J^{a-1}$. Then $X_1 Y m \in J^a m \subseteq J\mathfrak{K}(M)$ so that $X_1 Y m = X_1 m'$ for some $m' \in \mathfrak{K}(M)$ by the constant image property. It follows by Theorem 2.6 (ii) that $Ym - m' \in \mathfrak{K}(M)$. Hence $Ym \in \mathfrak{K}(M)$, showing that $J^{a-1}m \subseteq \mathfrak{K}(M)$.

Having established the claim, it follows by Lemmas 3.1 and 3.2 that

$$\frac{J^a\mathfrak{K}(M^*)}{J^b\mathfrak{K}(M^*)} \cong \frac{J^a(J\mathfrak{K}(M))^\perp}{J^b(J\mathfrak{K}(M))^\perp} = \frac{(J^{-a+1}\mathfrak{K}(M))^\perp}{(J^{-b+1}\mathfrak{K}(M))^\perp} \cong \left(\frac{J^{-b+1}\mathfrak{K}(M)}{J^{-a+1}\mathfrak{K}(M)}\right)^*. \qquad \square$$

4. Question 1.1 in the cases $p = 3$ and $J^2\mathfrak{K}(M) = 0$

In this section we identify two special cases in which Question 1.1 has an affirmative answer. We begin with a preliminary lemma.

Lemma 4.1. *If M is a kE-module of constant rank then*

$$\mathsf{Ker}(X_\alpha, M) \cap \mathsf{Im}(X_\alpha, M) = \mathsf{Ker}(X_\alpha, J^{-1}\mathfrak{K}(M)) \cap \mathsf{Im}(X_\alpha, J^{-1}\mathfrak{K}(M)).$$

Proof. The leftwards containment is clear, so let $m \in \mathsf{Ker}(X_\alpha, M) \cap \mathsf{Im}(X_\alpha, M)$. Since

$$\mathsf{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$$

by Theorem 2.6 (ii), we have $m \in \mathfrak{K}(M) \cap \mathsf{Im}(X_\alpha, M)$. Hence there exists $m' \in M$ such that $X_\alpha m' = m$. By Proposition 2.7 this implies $m' \in J^{-1}\mathfrak{K}(M)$ so that $m \in \mathsf{Im}(X_\alpha, J^{-1}\mathfrak{K}(M))$. This establishes the reverse containment. □

Proposition 4.2. *If M is a kE-module of constant rank, then for any $0 \neq \alpha \in \mathbb{A}^2(k)$, the number of blocks of length one in the Jordan type of X_α on $J^{-1}\mathfrak{K}(M)$ is equal to the number of blocks of length one in the Jordan type of X_α on M.*

Proof. By the lemma and its proof we have

$$\frac{\mathsf{Ker}(X_\alpha, M)}{\mathsf{Ker}(X_\alpha, M) \cap \mathsf{Im}(X_\alpha, M)} = \frac{\mathsf{Ker}(X_\alpha, J^{-1}\mathfrak{K}(M))}{\mathsf{Ker}(X_\alpha, J^{-1}\mathfrak{K}(M)) \cap \mathsf{Im}(X_\alpha, J^{-1}\mathfrak{K}(M))}.$$

The number of Jordan blocks of length one in the action of X_α on M is the dimension of the left hand term, and the number of Jordan blocks of length one in the action of X_α on $J^{-1}\mathfrak{K}(M)$ is the dimension of the right hand term. □

Theorem 4.3. *If $p = 3$ and M is a kE-module of constant Jordan type, then $J^{-i}\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$.*

Proof. If M is a $k(\mathbb{Z}/3)^2$-module of constant Jordan type, then $M = J^{-2}\mathfrak{K}(M)$ by Corollary 2.8. Hence we need only check that $J^{-1}\mathfrak{K}(M)$ has constant Jordan type. We fix $0 \neq \alpha \in \mathbb{A}^2(k)$ and let $[1]^a[2]^b[3]^c$ be the Jordan type of X_α on $J^{-1}\mathfrak{K}(M)$. Since the number of Jordan blocks in the action of X_α on $J^{-1}\mathfrak{K}(M)$ is equal to the number of Jordan blocks in the action of X_α on M by Theorem 2.6 (ii), we see that $a + b + c$ is independent of α. We also have $a + 2b + 3c = \dim_k J^{-1}\mathfrak{K}(M)$, which is clearly independent of α. Finally, a is equal to the number of Jordan blocks of length one in the action of X_α on M by Proposition 4.2. The fact that $J^{-1}\mathfrak{K}(M)$ has constant Jordan type now follows from the non-singularity of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix}.$$

□

In order to address the case in which $J^2\mathfrak{K}(M) = 0$, we make the following observation. Let M be any kE-module and let $n \geq 0$. For each non-zero $\alpha \in \mathbb{A}^2(k)$ there is a natural short exact sequence

$$(1) \qquad 0 \longrightarrow L_{n,\alpha} \longrightarrow \mathsf{Im}(X_\alpha^n, M/J^{j+1}\mathfrak{K}(M)) \longrightarrow \mathsf{Im}(X_\alpha^n, M/J^j\mathfrak{K}(M)) \longrightarrow 0$$

where

$$L_{n,\alpha} = \mathsf{Im}(X_\alpha^n, M/J^{j+1}\mathfrak{K}(M)) \cap J^j\mathfrak{K}(M)/J^{j+1}\mathfrak{K}(M).$$

If $M/J^{j+1}\mathfrak{K}(M)$ has constant Jordan type, then the dimension of the middle term is independent of α. We now introduce a criterion under which the dimensions of the outer two terms would also be independent of α.

Proposition 4.4. *Suppose $M/J^{j+1}\mathfrak{K}(M)$ has constant Jordan type and fix $n \geq 0$. If there exists a submodule N of $M/J^{j+1}\mathfrak{K}(M)$ such that $L_{n,\alpha} = \mathsf{Im}(X_\alpha^n, N)$ for all $0 \neq \alpha \in \mathbb{A}^2(k)$, then $\operatorname{rank}(X_\alpha^n, M/J^j\mathfrak{K}(M))$ is independent of α.*

Proof. Since $M/J^{j+1}\mathfrak{K}(M)$ has constant Jordan type, we see from the sequence (1) that

$$\operatorname{rank}(X_\alpha^n, N) + \operatorname{rank}(X_\alpha^n, M/J^j\mathfrak{K}(M))$$

is constant as a function of α. By Lemma 1.2 of [8], there exists a dense open subset $U \subseteq \mathbb{A}^2(k)$ such that $\alpha \in U$ if and only if $\operatorname{rank}(X_\alpha^n, M/J^j\mathfrak{K}(M))$ is maximal. Similarly, there exists a dense open subset $V \subseteq \mathbb{A}^2(k)$ such that $\alpha \in V$ if and only if $\operatorname{rank}(X_\alpha^n, N)$ is maximal. Since $\mathbb{A}^2(k)$ is irreducible, $U \cap V$ is non-empty. Choosing $\alpha \in U \cap V$, we simultaneously have rank$(X_\alpha^n, M/J^j\mathfrak{K}(M))$ maximal and rank(X_α^n, N) maximal. This forces both summands to be constant in α. $\square$

We now use the above theory to show that Question 1.1 is true in the special case where $J^2\mathfrak{K}(M) = 0$.

Theorem 4.5. *If M is a kE-module of constant Jordan type such that $J^2\mathfrak{K}(M) = 0$, then $J^{-i}\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$.*

Proof. Since $J^2\mathfrak{K}(M) = 0$, we have $M^* = J^{-1}\mathfrak{K}(M^*)$ by Theorem 3.3. We first show that $M^*/J^j\mathfrak{K}(M^*)$ has constant Jordan type for all $j \geq 0$ using downwards induction on j, beginning at $M^* = M^*/J^p\mathfrak{K}(M^*)$. So assume $M^*/J^{j+1}\mathfrak{K}(M^*)$ has constant Jordan type. Note for $0 \neq \alpha \in \mathbb{A}^2(k)$ that any Jordan block in the action of X_α on $M^*/J^j\mathfrak{K}(M^*)$ has length at most $j+1$. If $n \leq j$, let $m \in M^*/J^{j+1}\mathfrak{K}(M^*)$ such that $X_\alpha^n m \in J^j\mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)$. Since $\mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)$ has the constant image property, there exists $m' \in \mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)$ such that $X_\alpha^n m' = X_\alpha^n m$. Applying the sequence (1) to M^*, this yields

$$L_{n,\alpha} = \mathsf{Im}(X_\alpha^n, \mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)) \qquad \text{for all non-zero } \alpha.$$

It follows from Proposition 4.4 that rank$(X_\alpha^n, M^*/J^j\mathfrak{K}(M^*))$ is independent of α for all $n \leq j$. Letting $[1]^{a_1} \dots [j+1]^{a_{j+1}}$ denote the Jordan type of X_α on $\mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)$, we have

$$\begin{aligned} a_1 + 2a_2 + 3a_3 + \cdots + ja_j + (j+1)a_{j+1} &= \dim_k M^*/J^j\mathfrak{K}(M^*) \\ a_2 + 2a_3 + \cdots + (j-1)a_j + ja_{j+1} &= \operatorname{rank}(X_\alpha, M^*/J^j\mathfrak{K}(M^*)) \\ &\vdots \\ a_j + 2a_{j+1} &= \operatorname{rank}(X_\alpha^{j-1}, M^*/J^j\mathfrak{K}(M^*)) \\ a_{j+1} &= \operatorname{rank}(X_\alpha^j, M^*/J^j\mathfrak{K}(M^*)). \end{aligned}$$

Because the right hand terms are all independent of α, these equations have a solution for $a_1, \dots, a_{j+1}$ that is also independent of α. This shows that $M^*/J^j\mathfrak{K}(M^*)$ has constant Jordan type.

Having proved the claim, setting $j = -i + 1$, it now follows by another application of Theorem 3.3 that $J^{-i}\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$ as required. $\square$

5. A counterexample to Question 1.1

We now present an example of a kE-module M of constant Jordan type for which $J^{-1}\mathfrak{K}(M)$ does not have constant Jordan type. Let M be the 26 dimensional kE-module with basis

$a, b, c, \ldots, x, y, z$ on which g_1 acts via the matrix

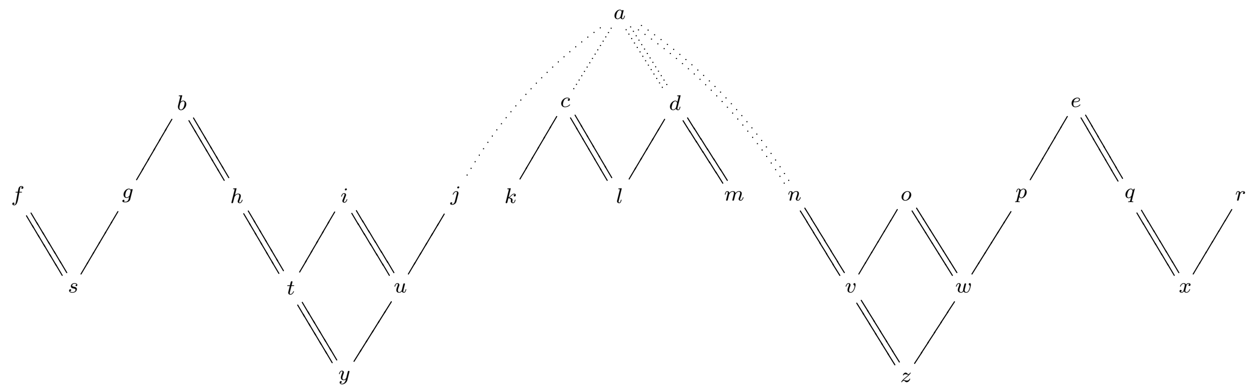
$$\left(\begin{array}{c|cccccccccccccccccccccccccc}
a &1 &&&&& &&&&& &&&&& &&&&& &&&&& \\
b && 1 &&&&& &&&&& &&&&& &&&&& &&&& \\
c &1 &&1 &&&&& &&&&& &&&&& &&&&& &&& \\
d &&&& 1 &&&&& &&&&& &&&&& &&&&& && \\
e &&&&& 1 &&&&& &&&&& &&&&& &&&&& & \\
f &&&&& & 1 &&&&& &&&&& &&&&& &&&&& \\
g && 1 &&&&& 1 &&&&& &&&&& &&&&& &&&& \\
h &&&&& &&& 1 &&&&& &&&&& &&&&& &&& \\
i &&&&& &&&& 1 &&&&& &&&&& &&&&& && \\
j &1 &&&&& &&&& 1 &&&&& &&&&& &&&&& & \\
k &&& 1 &&&&& &&& 1 &&&&& &&&&& &&&&& \\
l &&&& 1 &&&&& &&& 1 &&&&& &&&&& &&&& \\
m &&&&& &&&&& &&& 1 &&&&& &&&&& &&& \\
n &&&&& &&&&& &&&& 1 &&&&& &&&&& && \\
o &&&&& &&&&& &&&&& 1 &&&&& &&&&& & \\
p &&&&& 1 &&&&& &&&&& & 1 &&&&& &&&&& \\
q &&&&& &&&&& &&&&& && 1 &&&&& &&&& \\
r &&&&& &&&&& &&&&& &&& 1 &&&&& &&& \\
s &&&&& && 1 &&&&& &&&&& && 1 &&&&& && \\
t &&&&& &&&& 1 &&&&& &&&&& & 1 &&&&& & \\
u &&&&& &&&&& 1 &&&&& &&&&& & 1 &&&&& \\
v &&&&& &&&&& &&&&& 1 &&&&& && 1 &&&& \\
w &&&&& &&&&& &&&&& & 1 &&&&& && 1 &&& \\
x &&&&& &&&&& &&&&& &&& 1 &&&&& & 1 && \\
y &&&&& &&&&& &&&&& &&&&& & 1 &&&& 1 & \\
z &&&&& &&&&& &&&&& &&&&& &&& 1 &&& 1 \\
\hline
 & a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p & q & r & s & t & u & v & w & x & y & z
\end{array}\right)$$

and g_2 acts via the matrix

$$\left(\begin{array}{c|cccccccccccccccccccccccccc}
a &1 &&&&& &&&&& &&&&& &&&&& &&&&& \\
b && 1 &&&&& &&&&& &&&&& &&&&& &&&& \\
c &&& 1 &&&&& &&&&& &&&&& &&&&& &&& \\
d &1 &&& 1 &&&&& &&&&& &&&&& &&&&& && \\
e &&&&& 1 &&&&& &&&&& &&&&& &&&&& & \\
f &&&&& & 1 &&&&& &&&&& &&&&& &&&&& \\
g &&&&& && 1 &&&&& &&&&& &&&&& &&&& \\
h && 1 &&&&& & 1 &&&&& &&&&& &&&&& &&& \\
i &&&&& &&&& 1 &&&&& &&&&& &&&&& && \\
j &&&&& &&&&& 1 &&&&& &&&&& &&&&& & \\
k &&&&& &&&&& & 1 &&&&& &&&&& &&&&& \\
l &&& 1 &&&&& &&&& 1 &&&&& &&&&& &&&& \\
m &&&& 1 &&&&& &&&& 1 &&&&& &&&&& &&& \\
n &1 &&&&& &&&&& &&& 1 &&&&& &&&&& && \\
o &&&&& &&&&& &&&&& 1 &&&&& &&&&& & \\
p &&&&& &&&&& &&&&& & 1 &&&&& &&&&& \\
q &&&&& 1 &&&&& &&&&& && 1 &&&&& &&&& \\
r &&&&& &&&&& &&&&& &&& 1 &&&&& &&& \\
s &&&&& & 1 &&&&& &&&&& &&& 1 &&&&& && \\
t &&&&& &&& 1 &&&&& &&&&& && 1 &&&&& & \\
u &&&&& &&&& 1 &&&&& &&&&& && 1 &&&&& \\
v &&&&& &&&&& &&&& 1 &&&&& &&& 1 &&&& \\
w &&&&& &&&&& &&&&& 1 &&&&& &&& 1 &&& \\
x &&&&& &&&&& &&&&& && 1 &&&&& && 1 && \\
y &&&&& &&&&& &&&&& &&&&& 1 &&&&& 1 & \\
z &&&&& &&&&& &&&&& &&&&& && 1 &&&& 1 \\
\hline
 & a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p & q & r & s & t & u & v & w & x & y & z
\end{array}\right).$$

Here we have omitted entries equal to zero. This module structure is illustrated in the following diagram, where $X_1 = g_1 - 1$ acts via single edges downwards to the left, $X_2 = g_2 - 1$ acts via double edges downwards to the right, and the dotted edges indicate that $X_1.a = c + j$

and $X_2.a = d + n$.



We first show that M has constant Jordan type by verifying that $\operatorname{rank}(X_\alpha^n, M)$ is independent of $0 \neq \alpha \in \mathbb{A}^2(k)$ for $n = 1, 2, 3$. By inspection we see that $\operatorname{rank}(X_2, M) = 13$. For any $\lambda \in k$, the image of $X_1 + \lambda X_2$ on M is spanned by the elements

$$c + j + \lambda(d + n),\ g + \lambda h,\ k + \lambda l,\ l + \lambda m,\ p + \lambda q,\ s,\ t,\ u,\ v,\ w,\ x,\ y,\ z,$$

which are readily confirmed to be linearly independent. Hence $\operatorname{rank}(X_1 + \lambda X_2, M) = 13$ as well. Next observe that $\operatorname{rank}(X_2^2, M) = 5$, and for any $\lambda \in k$, the image of $(X_1 + \lambda X_2)^2$ on M is spanned by the linearly independent elements

$$k + u + 2\lambda l + \lambda^2(m + v),\ s + \lambda^2 t,\ w + \lambda^2 x,\ y,\ z$$

so that $\operatorname{rank}((X_1 + \lambda X_2)^2, M) = 5$. Finally, we have $\operatorname{rank}(X_2^3, M) = 2$, and for any $\lambda \in k$, the image of $(X_1 + \lambda X_2)^3$ on M is the submodule spanned by $\lambda^3 y$, $y + \lambda^3 z$ and z, which in any case has dimension two. Thus $\operatorname{rank}((X_1 + \lambda X_2)^3, M) = 2$ as required. This shows that M has constant Jordan type $[4]^2[3][2]^5[1]^5$.

To see that $J^{-1}\mathfrak{K}(M)$ does not have constant Jordan type, note that $\mathfrak{K}(M)$ is the submodule of M generated by

$$f,\ g,\ h,\ i,\ j,\ k,\ l,\ m,\ n,\ o,\ p,\ q,\ r$$

since this is the largest submodule of M having the constant image property. It follows that $J^{-1}\mathfrak{K}(M)$ is the submodule of M generated by

$$f,\ b,\ i,\ j,\ c,\ d,\ n,\ o,\ e,\ r.$$

Observe that $J^{-1}\mathfrak{K}(M)$ has a direct summand N generated by the elements f, b, i, j. Clearly N cannot have constant Jordan type because, for example, $X_2^3 N = \operatorname{span}_k(y)$, whereas $X_1^3 N = 0$. By Theorem 2.1 (ii), this shows that $J^{-1}\mathfrak{K}(M)$ cannot have constant Jordan type.

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