THE IP IS EQUIVALENT TO THE WOP

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Recall that the most general form of the induction principle (IP) is the statement

$$\forall x[\forall y(y < x \Rightarrow P(y)) \Rightarrow P(x)] \Longrightarrow \forall x\, P(x)$$

and the well ordering principle (WOP) is the statement

$$\exists x\, P(x) \Longrightarrow \exists x[P(x) \wedge \forall y(P(y) \Rightarrow x \leq y)].$$

Theorem. *The* IP *is equivalent to the* WOP.

Proof. For unary predicate F and binary predicate R, let G and S denote their respective negations. We have

$$\begin{aligned}
&\exists x\, F(x) \Longrightarrow \exists x[F(x) \wedge \forall y(F(y) \Rightarrow R(x,y))] \;\equiv \\
&\quad\equiv\; \neg\,\exists x[F(x) \wedge \forall y(F(y) \Rightarrow R(x,y))] \Longrightarrow \neg\,\exists x\, F(x) && \text{by contraposition} \\
&\quad\equiv\; \forall x\,\neg[F(x) \wedge \forall y(F(y) \Rightarrow R(x,y))] \Longrightarrow \forall x\, G(x) && \text{by moving } \neg \text{ across } \exists \\
&\quad\equiv\; \forall x[G(x) \vee \neg\,\forall y(F(y) \Rightarrow R(x,y))] \Longrightarrow \forall x\, G(x) && \text{by De Morgan's laws} \\
&\quad\equiv\; \forall x[\neg\,\forall y(F(y) \Rightarrow R(x,y)) \vee G(x)] \Longrightarrow \forall x\, G(x) && \text{by commutativity of } \vee \\
&\quad\equiv\; \forall x[\forall y(F(y) \Rightarrow R(x,y)) \Rightarrow G(x)] \Longrightarrow \forall x\, G(x) && \text{expressing } \Rightarrow \text{ via } \neg, \vee \\
&\quad\equiv\; \forall x[\forall y(S(x,y) \Rightarrow G(x)) \Rightarrow G(x)] \Longrightarrow \forall x\, G(x) && \text{by contraposition.}
\end{aligned}$$

Now let P be any unary predicate. Assume the IP. Substituting $\neg P$ into it yields

$$\forall x[\forall y(x < y \Rightarrow \neg P(x)) \Rightarrow \neg P(x)] \Longrightarrow \forall x\, \neg P(x).$$

Substituting $\leq$ for R and P for F in the above work, this is equivalent to

$$\exists x\, P(x) \Longrightarrow \exists x[P(x) \wedge \forall y(P(y) \Rightarrow x \leq y)],$$

thus the WOP is a theorem. Conversely, assume the WOP. Substituting $\neg P$ into it yields

$$\exists x\, \neg P(x) \Longrightarrow \exists x[\neg P(x) \wedge \forall y(\neg P(y) \Rightarrow x \leq y)].$$

Substituting $\neg P$ for F in the above work, this is equivalent to

$$\forall x[\forall y(x < y \Rightarrow \neg\neg P(x)) \Rightarrow \neg\neg P(x)] \Longrightarrow \forall x\, \neg\neg P(x),$$

which in turn is equivalent to

$$\forall x[\forall y(x < y \Rightarrow P(x)) \Rightarrow P(x)] \Longrightarrow \forall x\, P(x),$$

thus the IP is a theorem. □