$\mathsf{PA} - \mathsf{PMI} + \mathsf{WOP}$ IS NOT EQUIVALENT TO PA

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In this note we study a model for the system of logic that contains the axioms of Peano arithmetic, but with the principle of mathematical induction replaced by the well ordering principle. The construction turns out not to be a model for Peano arithmetic, disproving the often stated yet incorrect notion that the PMI and WOP are equivalent in the realm of Peano arithmetic. The model, as a poset having a zero object and successor function, was brought to the author's attention via a discussion at

`https://groups.google.com/forum/#!topic/sci.math/o5ILsxYtdHs`

The new contribution here is the introduction of 'addition' and 'multiplication' operations that respect the remaining Peano axioms, showing that the structure is a true model.

The underlying set in our model will be the disjoint union of two copies of the natural numbers

$$\mathbb{N} \sqcup \mathbb{N}.$$

If the natural number a lies in the first copy we write a_0, and we write a_1 otherwise. We take as our zero object the first copy of 0, that is, we set

$$\mathbf{0} = 0_0.$$

For convenience, we denote the successor of a natural number $a \in \mathbb{N}$ by a'. We define a successor function on $\mathbb{N} \sqcup \mathbb{N}$ by

$$\mathbf{S}(a_i) = a_i',$$

that is, $\mathbf{S}$ is the usual successor function on the respective copies of $\mathbb{N}$. The partial order $\preceq$ on $\mathbb{N} \sqcup \mathbb{N}$ is defined by

$$a_i \preceq b_j \qquad \text{if } i = 0 \text{ and } j = 1, \text{ or if } i = j \text{ and } a_i \leq b_i \text{ in } \mathbb{N}.$$

In other words, elements in the first copy of $\mathbb{N}$ always lie below those in the second copy, and elements that lie in the same copy of $\mathbb{N}$ are ordered as usual.

One readily verifies that the axiom schemata of Peano arithmetic involving only the zero object, successor function and partial ordering are all respected by this structure, i.e.,

$$\mathbf{S}(x) = \mathbf{S}(y) \implies x = y,$$

$$\neg\, \mathbf{0} = \mathbf{S}(x),$$

$$x \preceq \mathbf{0} \iff x = \mathbf{0},$$

$$x \preceq \mathbf{S}(y) \iff (x \preceq y \,\vee\, x = \mathbf{S}(y)),$$

$$x \preceq y \,\vee\, y \preceq x$$

all have true interpretations.

It is now clear that the WOP holds in this structure but the PMI does not. Indeed, the first copy of $\mathbb{N}$ satisfies the hypotheses of the PMI, yet is not all of $\mathbb{N} \sqcup \mathbb{N}$.

Our next goal is to show that this structure may be endowed with addition and multiplication rules that respect the remaining Peano axiom schemata. We define

$$a_i + b_j = (a+b)_i \qquad \text{and} \qquad a_i \times b_j = (ab)_j.$$

Roughly speaking, addition is dominated by the left hand term, whereas multiplication is dominated by the right hand term. We now confirm that the remaining axiom schemata

$$\begin{gathered} x + \mathbf{0} = x, \\ x + \mathbf{S}(y) = \mathbf{S}(x+y), \\ x \times \mathbf{0} = \mathbf{0}, \\ x \times \mathbf{S}(y) = (x \times y) + x \end{gathered}$$

of Peano arithmetic are satisfied. Indeed, in our structure one computes that

$$\begin{gathered} a_i + \mathbf{0} = a_i + 0_0 = (a+0)_i = a_i, \\ a_i + \mathbf{S}(b_j) = a_i + b'_j = (a+b')_i = ((a+b)')_i = \mathbf{S}((a+b)_i) = \mathbf{S}(a_i + b_j), \\ a_i \times \mathbf{0} = a_i \times 0_0 = (a0)_0 = 0_0 = \mathbf{0}, \\ a_i \times \mathbf{S}(b_j) = a_i \times b'_j = (ab')_j = (ab+a)_j = (ab)_j + a_i = (a_i \times b_j) + a_i. \end{gathered}$$

We remark that the non-commutativity of these operations is somewhat expected, as the associative and commutative laws of addition and multiplication in $\mathbb{N}$ are consequences of the PMI, the latter having been discarded.

Finally, we recall that in the presence of the axiom schema

$$x = \mathbf{0} \vee \exists y \, (x = \mathbf{S}(y))$$

of Robinson arithmetic (Q), the WOP *does* imply the PMI. It would therefore be accurate to say that

$$\mathsf{Q} + \mathsf{PMI} \textit{ is equivalent to } \mathsf{Q} + \mathsf{WOP}.$$

Incidentally, because our model witnesses that $\mathsf{PA} - \mathsf{PMI} + \mathsf{WOP}$ is not Peano arithmetic, the above axiom schema must have a false interpretation in it. Indeed, 0_1 is non-zero but is not the successor of any element under $\mathbf{S}$.