

MODULES OF CONSTANT JORDAN TYPE, PULLBACKS OF BUNDLES AND GENERIC KERNEL FILTRATIONS

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ABSTRACT. Let kE denote the group algebra of an elementary abelian p -group of rank r over an algebraically closed field of characteristic p . We investigate the functors \mathcal{F}_i from kE -modules of constant Jordan type to vector bundles on $\mathbb{P}^{r-1}(k)$, constructed by Benson and Pevtsova. For a kE -module M of constant Jordan type, we show that restricting the sheaf $\mathcal{F}_i(M)$ to a dimension $s - 1$ linear subvariety of $\mathbb{P}^{r-1}(k)$ is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying \mathcal{F}_i .

In the case $r = 2$, we examine the generic kernel filtration of M in order to show that $\mathcal{F}_i(M)$ may be computed on certain subquotients of M whose Loewy lengths are bounded in terms of i . More precise information is obtained by applying similar techniques to the n th power generic kernel filtration of M . The latter approach also allows us to generalise our results to higher ranks r .

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1. INTRODUCTION

The goal of this paper is to further investigate a curious functorial relationship between the category of finitely generated kE -modules and the category of coherent sheaves on the projective space $\mathbb{P}^{r-1}(k)$, where E is an elementary abelian p -group of rank r and k is an

algebraically closed field of characteristic p . Specifically, we wish to better understand the functors

$$\mathcal{F}_i: \mathbf{mod}(kE) \longrightarrow \mathbf{coh}(\mathbb{P}^{r-1}(k)), \quad 1 \leq i \leq p$$

introduced by Benson and Pevtsova [4]. Interest in the functors \mathcal{F}_i originated in the study of kE -modules of constant Jordan type, which were defined by Carlson, Friedlander and Pevtsova [5]. Denoting the subcategory of modules of constant Jordan type by $\mathbf{cJt}(kE)$, Benson and Pevtsova showed that the functors \mathcal{F}_i descend to functors

$$\mathcal{F}_i: \mathbf{cJt}(kE) \longrightarrow \mathbf{vec}(\mathbb{P}^{r-1}(k)),$$

where $\mathbf{vec}(\mathbb{P}^{r-1}(k))$ is the category of vector bundles on $\mathbb{P}^{r-1}(k)$. We remark that neither of the latter two categories is well understood. Whereas the study of modules of constant Jordan type is a relatively new enterprise, the attempt to understand what sorts of vector bundles can live on $\mathbb{P}^{r-1}(k)$ has been ongoing since the advent of modern algebraic geometry, and with limited success. Accordingly, a thorough understanding of the functors \mathcal{F}_i should be of interest to representation theorists and algebraic geometers alike.

In this direction, our aim is to further establish some sort of dictionary between modules of constant Jordan type and vector bundles on $\mathbb{P}^{r-1}(k)$ via the functors \mathcal{F}_i . For example, one of the common techniques of the algebraic geometer is that of restricting a vector bundle on $\mathbb{P}^{r-1}(k)$ to a line L in $\mathbb{P}^{r-1}(k)$ in order to compute its so called ‘splitting type’. Any such closed immersion $L \subseteq \mathbb{P}^{r-1}(k)$ is obtained by applying the \mathbf{Proj} functor to a surjective homogeneous ring homomorphism

$$k[Y_1, \dots, Y_r] \longrightarrow k[Z_1, Z_2].$$

In Section 3 we generalise this situation a bit and show that any surjective ring homomorphism of the form

$$k[Y_1, \dots, Y_r] \longrightarrow k[Z_1, \dots, Z_s], \quad s \leq r$$

arises naturally from what we call a *homogeneously embedded s -shifted subgroup* kE' of kE . Our main result related to this is that, under the functors \mathcal{F}_i , pulling back along such a closed immersion $\mathbb{P}^{s-1}(k) \hookrightarrow \mathbb{P}^{r-1}(k)$ corresponds to restricting scalars along the inclusion $kE' \hookrightarrow kE$. Specifically, we obtain the following.

Theorem. *Let kE' be a homogeneously embedded s -shifted subgroup of kE and let*

$$f: \mathbb{P}^{s-1}(k) \longrightarrow \mathbb{P}^{r-1}(k)$$

be the corresponding closed immersion. If M is a kE -module of constant Jordan type, then for all $1 \leq i \leq p$ we have $f^ \mathcal{F}_i(M) \cong \mathcal{F}_i(M \downarrow_{kE'})$.*

Here, the notation has been abused slightly so that \mathcal{F}_i denotes both functors on $\mathbf{mod}(kE)$ and $\mathbf{mod}(kE')$, respectively.

In connection with the long term goal of studying kE -modules of constant Jordan type by looking at splitting types of vector bundles on $\mathbb{P}^{r-1}(k)$, the remainder of this paper will be dedicated to better understanding the behaviour of modules of constant Jordan type in the case $r = 2$, so that the bundles $\mathcal{F}_i(M)$ live over $\mathbb{P}^1(k)$. The representation theory of kE

in the case $r = 2$ was closely investigated by Carlson, Friedlander and Suslin [6]. In that paper, the authors constructed an interesting functorial invariant of a kE -module called the *generic kernel*. If M is any finite dimensional kE -module in rank two, then its generic kernel $\mathfrak{K}(M)$ can be characterised as the largest submodule of M having the so called ‘equal images property’. (See Definition 5.1.) The generic kernel gives rise to a filtration of M whose terms are $J^i\mathfrak{K}(M)$, where $J = J(kE)$ denotes the Jacobson radical of kE , and for $j > 0$, $J^{-j}\mathfrak{K}(M)$ denotes the collection of elements $m \in M$ for which $J^j m \subseteq \mathfrak{K}(M)$. In the first author’s [1], this was called the *generic kernel filtration*.

If M has something called the ‘constant rank’ property, which is a relatively mild condition, then the generic kernel filtration of M has some interesting features. For example, its filtered quotients are semisimple, and there is a well behaved duality theory. (See Lemma 7.7.) Our results here will show further that for a kE -module M of constant Jordan type, the various layers of the generic kernel filtration allow one to compute the vector bundles $\mathcal{F}_i(M)$ on what are generally much smaller subquotients of M . In their fullest, our results establish the following.

Theorem. *If $r = 2$ and M is a kE -module of constant Jordan type, then for each $1 \leq i \leq p$ we have*

$$\mathcal{F}_i(M) \cong \mathcal{F}_i(J^{-i}\mathfrak{K}(M)/J^{i+1}\mathfrak{K}(M)).$$

Our main objective in reducing to the above subquotients when computing $\mathcal{F}_i(M)$ was the following: Although the vector bundles on $\mathbb{P}^1(k)$ may be described succinctly (every bundle is a direct sum of twists of the structure sheaf), the category of kE -modules having constant Jordan type is known to be wild unless $p = 2$. It would be satisfying to find a subcategory of $\text{mod}(kE)$ through which the functor \mathcal{F}_i factors, one which lends itself to some sort of structure theorem. Unfortunately, even when computing $\mathcal{F}_1(M)$, the above theorem deals with the subquotient $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. Although such subquotients have Loewy length only three, the class of modules of the form $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ (where M has constant Jordan type) remains wild. There must certainly be a better behaved structural invariant of M that determines what $\mathcal{F}_i(M)$ is. In other words, the goal of our program is to determine exactly how much of the structure of M the functors \mathcal{F}_i actually detect.

The search for such a structural invariant has led us to investigate a different filtration, namely the n th power generic kernel filtration. As with the generic kernel, the n th power generic kernel ($n \geq 1$) of a kE -module M was also introduced in [6]. We will show that the n th power generic kernels, along with their duals, give rise to a filtration of M that performs the same task with respect to computing $\mathcal{F}_i(M)$ as the regular generic kernel filtration does, but with two added benefits: First, a suitable choice of definition allows us to generalise our results to the case $r > 2$, and second, the Loewy length three subquotients in the n th power generic kernel filtration on which $\mathcal{F}_1(M)$ is computed appear to have more tractable structures. In particular, it appears that $\mathcal{F}_1(M)$ is determined by the Loewy length two summands of these subquotients. We make a precise conjecture about this point in our final section.

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2. BACKGROUND ON ELEMENTARY ABELIAN p -GROUP REPRESENTATIONS

Let p be a prime number and k an algebraically closed field of characteristic p . Throughout this paper, $E \cong (\mathbb{Z}/p)^r$ will denote an elementary abelian p -group of rank r . In what follows, we choose a collection of pairwise commuting generators g_1, \dots, g_r of E . Letting X_i denote the element $g_i - 1$ in the group algebra kE , we have $X_i^p = (g_i - 1)^p = 0$ since k has characteristic p . We may therefore identify kE with the truncated polynomial ring $k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$. In particular, kE is a local ring, and the Jacobson radical of kE is generated by the elements X_i .

If $\alpha = (\lambda_1, \dots, \lambda_r)$ is any point in the affine space $\mathbb{A}^r(k)$, we define the element

$$X_\alpha = \lambda_1 X_1 + \dots + \lambda_r X_r \in kE.$$

Observe that we have $X_\alpha^p = 0$ for all α . The assignment $\alpha \mapsto X_\alpha$ allows one to identify $\mathbb{A}^r(k)$ with $\text{Rad}(kE)/\text{Rad}^2(kE)$. Note that for each non-zero α , the element $1 + X_\alpha \in kE$ has multiplicative order p , hence it generates a subgroup of kE^\times isomorphic to \mathbb{Z}/p . The subalgebra $k\langle 1 + X_\alpha \rangle \subseteq kE$ is called a *cyclic shifted subgroup* of kE . The nomenclature is designed to indicate that, although $k\langle 1 + X_\alpha \rangle$ is a group algebra, $\langle 1 + X_\alpha \rangle$ is not a subgroup of E in general.

Throughout this paper we will deal exclusively with finitely generated (i.e., finite dimensional) kE -modules. If M is a kE -module and α is a point in $\mathbb{A}^r(k)$, then because $X_\alpha^p = 0$, the Jordan canonical form of the matrix representing the action of X_α on M consists of Jordan blocks whose eigenvalues are all zero and whose lengths are at most p . The *Jordan type* of X_α on M is defined to be the partition

$$\text{JType}(X_\alpha, M) = [p]^{a_p} [p-1]^{a_{p-1}} \dots [1]^{a_1}$$

of $\dim_k(M)$, where X_α acts on M via a_j Jordan blocks of length j . Recall that representations of $k(\mathbb{Z}/p)$ are classified in terms of Jordan canonical forms. In that context, if $\alpha \neq 0$, then the Jordan type of X_α on M is precisely the isomorphism type of $M \downarrow_{k\langle 1 + X_\alpha \rangle}$ when viewed as a $k(\mathbb{Z}/p)$ -module via the identification $\langle 1 + X_\alpha \rangle \cong \mathbb{Z}/p$.

At first, one might be tempted to try to classify the indecomposable objects in $\text{mod}(kE)$. Unfortunately, kE has wild representation type unless $r = 1$ or $r = p = 2$, which essentially makes that task impossible. We therefore confine ourselves to identifying special subcategories of $\text{mod}(kE)$ that we hope to better understand in terms of certain invariants. This paper is primarily concerned with the category of modules of constant Jordan type, which were introduced by Carlson, Friedlander and Pevtsova [5].

Definition 2.1. A kE -module M has *constant Jordan type* if the partition $\text{JType}(X_\alpha, M)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$. If M has constant Jordan type and $\text{JType}(X_\alpha, M) = [p]^{a_p} \dots [1]^{a_1}$ for all non-zero α , then we say that M has *constant Jordan type* $[p]^{a_p} \dots [1]^{a_1}$. We denote the full subcategory of modules of constant Jordan type by $\text{cJt}(kE)$.

For our purposes, the main feature of modules of constant Jordan type is that they give rise to vector bundles (i.e., locally free coherent sheaves) on $\mathbb{P}^{r-1}(k)$ in a natural way. Let V be the subspace of kE spanned by X_1, \dots, X_r . For $1 \leq i \leq r$, let $Y_i \in V^\#$ be the basis

element dual to X_i . The Y_i then act as homogeneous coordinate functions on V , and we identify $\mathbb{P}^{r-1}(k)$ with $\text{Proj } k[Y_1, \dots, Y_r]$. For an arbitrary kE -module M , let \widetilde{M} denote the coherent sheaf $M \otimes_k \mathcal{O}_{\mathbb{P}^{r-1}}$. In [7], Friedlader and Pevtsova introduced the operators

$$\theta_M: \widetilde{M}(n) \longrightarrow \widetilde{M}(n+1)$$

defined locally as follows: Any section of $\widetilde{M}(n)$ is of the form $m \otimes f$, where $m \in M$ and f is a homogeneous rational function of degree n in Y_1, \dots, Y_r . The map θ_M is defined by mapping $m \otimes f$ to the section $\sum_i X_i m \otimes Y_i f$ of $\widetilde{M}(n+1)$. The virtue of this setup is that if $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ is a closed point and $\alpha \in \mathbb{A}^r(k)$ is a point lying above $\bar{\alpha}$, then the fibre of θ_M at $\bar{\alpha}$ recovers (up to a scalar factor) the k -linear map $X_\alpha: M \rightarrow M$.

We now describe the functors of interest in this paper. For a kE -module M and $1 \leq i \leq p$, Benson and Pevtsova [4] defined the coherent sheaves

$$\mathcal{F}_i(M) = \frac{\text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}}{\text{Ker } \theta_M \cap \text{Im } \theta_M^i}.$$

Here, $\text{Ker } \theta_M$ denotes the kernel of the morphism $\widetilde{M} \rightarrow \widetilde{M}(1)$, whereas, for $j = i - 1$ and i , $\text{Im } \theta_M^j$ denotes the image of the morphism $\widetilde{M}(-j) \rightarrow \widetilde{M}$. With these conventions, $\mathcal{F}_i(M)$ is a subquotient of \widetilde{M} . The following, which appeared in [4], is the main fact concerning the functors \mathcal{F}_i .

Proposition 2.2. *A kE -module M has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ if and only if, for each $1 \leq i \leq p$, the coherent sheaf $\mathcal{F}_i(M)$ is a vector bundle of rank a_i on $\mathbb{P}^{r-1}(k)$.*

The main result of [4] showed that the functor \mathcal{F}_1 realises all vector bundles on $\mathbb{P}^{r-1}(k)$ up to a Frobenius twist. We record it here in order to motivate our overall interest in the functors \mathcal{F}_i , and our particular interest in the behaviour of the functor \mathcal{F}_1 , the latter being the subject of our examples.

Theorem 2.3. *If $p = 2$ and \mathcal{F} is a vector bundle of rank s on $\mathbb{P}^{r-1}(k)$, then there exists a kE -module of constant Jordan type of the form $[p]^n [1]^s$ (for some n) such that $\mathcal{F}_1(M) \cong \mathcal{F}$. If $p > 2$ and \mathcal{F} is a vector bundle on $\mathbb{P}^{r-1}(k)$, then there exists a kE -module of the form $[p]^n [1]^s$ such that $\mathcal{F}_1(M) \cong F^* \mathcal{F}$, where $F: \mathbb{P}^{r-1}(k) \rightarrow \mathbb{P}^{r-1}(k)$ is the Frobenius morphism.*

3. PULLBACKS OF BUNDLES AND HOMOGENEOUSLY EMBEDDED SUBGROUPS

The natural generalisation of a cyclic shifted subgroup of kE is a *rank s shifted subgroup* of kE , where $s \leq r$ is a fixed positive integer. Specifically, a rank s shifted subgroup is a subalgebra of kE that is isomorphic to the group algebra kE' , where E' is an elementary abelian p -group of rank s . Any embedding $\phi: kE' \hookrightarrow kE$ is obtained by mapping a choice of generators T_1, \dots, T_s of $\text{Rad}(kE')$ to elements $\phi(T_1), \dots, \phi(T_s) \in \text{Rad}(kE)$ whose images in $\text{Rad}(kE)/\text{Rad}^2(kE)$ are linearly independent. In the following, we restrict attention to the embeddings ϕ for which the elements $\phi(T_j)$ are linear combinations of the generators X_1, \dots, X_r of $\text{Rad}(kE)$. We call such embeddings *homogeneously embedded s -shifted subgroups*. As we shall see, homogeneously embedded s -shifted subgroups give rise to closed

immersions $\mathbb{P}^{s-1} \hookrightarrow \mathbb{P}^{r-1}$ that are of interest in the study of vector bundles on projective space, e.g., embeddings of lines into \mathbb{P}^{r-1} .

Following the notation of the previous section, we continue to let V denote the subspace of kE spanned by X_1, \dots, X_r , and we let U be the subspace of kE' spanned by T_1, \dots, T_s . By definition, the homogeneous embedding $\phi: kE' \hookrightarrow kE$ is given by a linear embedding $U \hookrightarrow V$, which is represented by an $r \times s$ matrix $\mathbf{A} = (a_{ij})$. Specifically, we have $\phi(T_j) = \sum_{i=1}^r a_{ij} X_i$. Taking k -linear duals, the matrix \mathbf{A}^t induces a surjective linear map $V^\# \rightarrow U^\#$. Letting Z_1, \dots, Z_s denote the dual elements in $U^\#$ that correspond to T_1, \dots, T_s , respectively, \mathbf{A}^t then gives rise to a surjective graded homomorphism of k -algebras

$$\phi^\#: k[Y_1, \dots, Y_r] \longrightarrow k[Z_1, \dots, Z_s].$$

Specifically, we have $\phi^\#(Y_i) = \sum_{j=1}^s a_{ij} Z_j$. Finally, applying the functor \mathbf{Proj} , the graded homomorphism $\phi^\#$ induces the desired closed immersion $f: \mathbb{P}^{s-1} \hookrightarrow \mathbb{P}^{r-1}$.

For a finite dimensional kE -module M , we now wish to compare the coherent sheaves $f^*(\mathcal{F}_i(M))$ and $\mathcal{F}_i(M \downarrow_{kE'})$ on \mathbb{P}^{s-1} , where by abuse of notation, \mathcal{F}_i denotes both functors $\mathbf{mod}(kE) \rightarrow \mathbf{coh}(\mathbb{P}^{r-1})$ and $\mathbf{mod}(kE') \rightarrow \mathbf{coh}(\mathbb{P}^{s-1})$, respectively.

Proposition 3.1. *Let M be any finite dimensional kE -module. Let \mathcal{E}^\bullet denote the sequence of coherent sheaves*

$$\dots \longrightarrow \widetilde{M}(n-1) \xrightarrow{\theta_M} \widetilde{M}(n) \xrightarrow{\theta_M} \widetilde{M}(n+1) \longrightarrow \dots$$

on \mathbb{P}^{r-1} and let $\mathcal{E} \downarrow_{kE'}^\bullet$ denote the sequence of coherent sheaves

$$\dots \longrightarrow \widetilde{M \downarrow_{kE'}}(n-1) \xrightarrow{\theta_{M \downarrow_{kE'}}} \widetilde{M \downarrow_{kE'}}(n) \xrightarrow{\theta_{M \downarrow_{kE'}}} \widetilde{M \downarrow_{kE'}}(n+1) \longrightarrow \dots$$

on \mathbb{P}^{s-1} . (Observe that these are not chain complexes unless $p = 2$.) Then $f^*\mathcal{E}^\bullet$ is naturally isomorphic to $\mathcal{E} \downarrow_{kE'}^\bullet$ in the functor category $\mathbf{Fun}(\mathbb{Z}, \mathbf{coh}(\mathbb{P}^{s-1}))$.

Proof. For $n \in \mathbb{Z}$, note that $\widetilde{M}(n) = M \otimes_k \mathcal{O}_{\mathbb{P}^{r-1}}(n)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{r-1}}(n)^{\oplus d}$, where $d = \dim_k M$. In this way, $\theta_M = \sum_i X_i \otimes Y_i$ can be viewed as the $d \times d$ matrix $\sum_i Y_i X_i$ with entries in $k[Y_1, \dots, Y_r]$. It follows that $f^*\theta_M$ is the matrix map

$$\mathcal{O}_{\mathbb{P}^{s-1}}(n)^{\oplus d} = f^*(\mathcal{O}_{\mathbb{P}^{r-1}}(n)^{\oplus d}) \xrightarrow{\sum_i \phi^\#(Y_i) X_i} f^*(\mathcal{O}_{\mathbb{P}^{r-1}}(n+1)^{\oplus d}) = \mathcal{O}_{\mathbb{P}^{s-1}}(n+1)^{\oplus d}.$$

On the other hand, $\widetilde{M \downarrow_{kE'}}(n) = M \downarrow_{kE'} \otimes_k \mathcal{O}_{\mathbb{P}^{s-1}}(n)$ can be identified with $\mathcal{O}_{\mathbb{P}^{s-1}}(n)^{\oplus d}$, hence we may view $\theta_{M \downarrow_{kE'}} = \sum_j T_j \otimes Z_j$ as a $d \times d$ matrix with entries in $k[Z_1, \dots, Z_s]$. But as a k -linear endomorphism of M , T_j acts via the embedding $\phi: kE' \rightarrow kE$. In other words, $\theta_{M \downarrow_{kE'}}$ acts via the matrix

$$\sum_j Z_j \phi(T_j) = \sum_{i,j} Z_j (a_{ij} X_i) = \sum_{i,j} (a_{ij} Z_j) X_i = \sum_i \phi^\#(Y_i) X_i.$$

Taking the vertical arrows in

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^{s-1}}(n)^{\oplus d} & \xrightarrow{\sum_i \phi^{\#(Y_i)X_i}} & \mathcal{O}_{\mathbb{P}^{s-1}}(n+1)^{\oplus d} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{P}^{s-1}}(n)^{\oplus d} & \xrightarrow{\sum_j Z_j \phi(T_j)} & \mathcal{O}_{\mathbb{P}^{s-1}}(n+1)^{\oplus d} \end{array}$$

to be the $d \times d$ identity matrix induces the required isomorphism $f^* \mathcal{E}^\bullet \xrightarrow{\sim} \mathcal{E} \downarrow_{kE'}$. \square

Corollary 3.2. *If M is any finite dimensional kE -module, then*

$$\mathcal{F}_i(M \downarrow_{kE'}) \cong \frac{\text{Ker}(f^* \theta_M) \cap \text{Im}(f^* \theta_M^{i-1})}{\text{Ker}(f^* \theta_M) \cap \text{Im}(f^* \theta_M^i)}.$$

Before giving our next result, we first recall an apparently standard lemma whose proof we provide for completeness.

Lemma 3.3. *Let $f: X \rightarrow Y$ be any morphism of locally ringed spaces. If*

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$$

is a short exact sequence of \mathcal{O}_Y -modules with \mathcal{G}'' locally free, then

$$0 \longrightarrow f^* \mathcal{G}' \longrightarrow f^* \mathcal{G} \longrightarrow f^* \mathcal{G}'' \longrightarrow 0$$

is a short exact sequence of \mathcal{O}_X -modules.

Proof. The functor f^* is right exact, so we have an exact sequence of \mathcal{O}_X -modules

$$L_1 f^* \mathcal{G}'' \longrightarrow f^* \mathcal{G}' \longrightarrow f^* \mathcal{G} \longrightarrow f^* \mathcal{G}'' \longrightarrow 0.$$

Note that the leftmost term is

$$L_1((-\otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \circ f^{-1})(\mathcal{G}'') = L_1(-\otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)(f^{-1}\mathcal{G}'') = \text{Tor}_1^{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}'', \mathcal{O}_X),$$

where the first equality holds since f^{-1} is exact. Now let $x \in X$ and consider the localisation $(L_1 f^* \mathcal{G}'')_x$. Because Tor commutes with localisation, this is equal to

$$\text{Tor}_1^{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}'', \mathcal{O}_X)_x = \text{Tor}_1^{\mathcal{O}_{Y,f(x)}}(\mathcal{G}''_{f(x)}, \mathcal{O}_{X,x}).$$

Since \mathcal{G}'' is locally free, the right hand term is zero, hence $(L_1 f^* \mathcal{G}'')_x = 0$. This being true for all $x \in X$, it follows that $L_1 f^* \mathcal{G}'' = 0$. \square

Returning to the case where $f: \mathbb{P}^{s-1} \rightarrow \mathbb{P}^{r-1}$ is the closed immersion corresponding to the embedding $\phi: kE' \rightarrow kE$, we are now in a position to prove the following.

Theorem 3.4. *If M is a kE -module of constant Jordan type, then $f^* \mathcal{F}_i(M) \cong \mathcal{F}_i(M \downarrow_{kE'})$ for all $1 \leq i \leq p$.*

Proof. We first claim that we can identify $f^*(\mathrm{Im} \theta_M^i)$ and $\mathrm{Im}(f^*\theta_M^i)$ as subobjects of $f^*\widetilde{M}$ for all $1 \leq i \leq p$. Since M has constant Jordan type, the morphism $\theta_M^i: \widetilde{M}(-i) \rightarrow \widetilde{M}$ has locally free cokernel. By Lemma 3.3, applying f^* gives an exact sequence

$$0 \longrightarrow f^*(\mathrm{Im} \theta_M^i) \longrightarrow f^*\widetilde{M} \longrightarrow f^*(\mathrm{Coker} \theta_M^i) \longrightarrow 0$$

By the universal property of cokernels, there is a unique isomorphism u making the following diagram commute.

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^*(\mathrm{Im} \theta_M^i) & \xrightarrow{f^*\theta_M^i} & f^*\widetilde{M} & \xrightarrow{q'} & f^*(\mathrm{Coker} \theta_M^i) \longrightarrow 0 \\ & & & & \searrow q & & \uparrow u \\ & & & & & & \mathrm{Coker}(f^*\theta_M^i) \end{array}$$

Since u is an isomorphism, we have $\mathrm{Ker} q' = \mathrm{Ker} q$, hence $f^*(\mathrm{Im} \theta_M^i) = \mathrm{Im}(f^*\theta_M^i)$ as subobjects of $f^*\widetilde{M}$. Now consider the short exact sequence

$$0 \longrightarrow \mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^{i-1} \longrightarrow \mathrm{Im} \theta_M^{i-1} \xrightarrow{\theta_M} \mathrm{Im} \theta_M^i(1) \longrightarrow 0$$

of coherent sheaves on \mathbb{P}^{r-1} . Since M has constant Jordan type, $\mathrm{Im} \theta_M^i$ is locally free, so a similar argument shows that $f^*(\mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^{i-1}) = \mathrm{Ker}(f^*\theta_M) \cap \mathrm{Im}(f^*\theta_M^{i-1})$ as subobjects of $f^*\widetilde{M}$ for all i .

Finally, consider the short exact sequence

$$0 \longrightarrow \mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^i \longrightarrow \mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^{i-1} \longrightarrow \mathcal{F}_i(M) \longrightarrow 0$$

defining $\mathcal{F}_i(M)$. Because M has constant Jordan type, Proposition 2.1 of [4] tells us that $\mathcal{F}_i(M)$ is locally free. Another application of Lemma 3.3 then reveals that

$$0 \longrightarrow f^*(\mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^i) \longrightarrow f^*(\mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^{i-1}) \longrightarrow f^*\mathcal{F}_i(M) \longrightarrow 0$$

is exact. In light of Corollary 3.2, the diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^*(\mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^i) & \longrightarrow & f^*(\mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^{i-1}) & \longrightarrow & f^*\mathcal{F}_i(M) \longrightarrow 0 \\ & & \parallel & & \parallel & & \\ 0 & \longrightarrow & \mathrm{Ker}(f^*\theta_M) \cap \mathrm{Im}(f^*\theta_M^i) & \longrightarrow & \mathrm{Ker}(f^*\theta_M) \cap \mathrm{Im}(f^*\theta_M^{i-1}) & \longrightarrow & \mathcal{F}_i(M \downarrow_{kE'}) \longrightarrow 0 \end{array}$$

immediately implies that $f^*\mathcal{F}_i(M) \cong \mathcal{F}_i(M \downarrow_{kE'})$. \square

Remark 3.5. A similar statement to Theorem 3.4 cannot hold for the pushforward along f . In particular, $f_*\mathcal{F}_i(M \downarrow_{kE'})$ is never isomorphic to $\mathcal{F}_i(M)$ unless the latter sheaf is zero or $s = r$. This is because the pushforward of a sheaf along a closed immersion that is not surjective is never globally supported, hence cannot be locally free of non-zero rank.

4. MORE ON THE OPERATOR θ_M AND VECTOR BUNDLES

This section consists of extensions of the preliminary results in [4]. We give an explicit exposition of these key ideas, as they will be used extensively throughout the sequel. We first recall what is perhaps the most important fact about locally free sheaves on projective space. It is traditionally referenced as Exercise II.5.8 of [11].

Lemma 4.1. *Let X be a reduced connected noetherian scheme and $f: \mathcal{E} \rightarrow \mathcal{E}'$ a morphism of locally free sheaves on X . Then the dimension of the fibre*

$$f \otimes k(x): \mathcal{E} \otimes_{\mathcal{O}_X} k(x) \longrightarrow \mathcal{E}' \otimes_{\mathcal{O}_X} k(x)$$

is independent of $x \in X$ if and only if $\text{Coker } f$ is locally free.

If these conditions hold, then the coherent sheaf $\text{Im } f$ is also locally free.

Proof. The second statement follows from the short exact sequence

$$0 \longrightarrow \text{Im } f \longrightarrow \mathcal{E}' \longrightarrow \text{Coker } f \longrightarrow 0$$

in which the map on the right is a surjection of locally free sheaves. □

Notation 4.2. Observe that if M is a kE -module, then for each non-zero $\alpha = (\lambda_1, \dots, \lambda_r)$ in $\mathbb{A}^r(k)$, the submodules $\text{Im}(X_\alpha^i, M)$ and $\text{Ker}(X_\alpha^i, M)$ are uniquely determined by the class $\bar{\alpha} = [\lambda_1 : \dots : \lambda_r]$ in $\mathbb{P}^{r-1}(k)$. In what follows, we shall often find it convenient to use the closed point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ to parameterise the action of the non-zero element X_α on M .

The following two lemmas were instrumental in the proof of Proposition 2.1 of [4]. We provide details here, because the same reasoning will be used later when we examine the behaviour of vector bundles with respect to submodules.

Lemma 4.3. *Let M be a kE -module and suppose that the rank of X_α^i acting as a k -linear endomorphism of M is independent of the choice of $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ for some $i \geq 0$. Then the coherent sheaf $\text{Im } \theta_M^i$ is locally free. Moreover, the fibre of the short exact sequence*

$$0 \longrightarrow \text{Im } \theta_M^i \longrightarrow \widetilde{M} \longrightarrow \text{Coker } \theta_M^i \longrightarrow 0$$

at a point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ may be identified with the natural short exact sequence

$$0 \longrightarrow \text{Im}(X_\alpha^i, M) \longrightarrow M \longrightarrow \text{Coker}(X_\alpha^i, M) \longrightarrow 0.$$

Proof. Recall that the fibre of $\widetilde{M}(-i) \xrightarrow{\theta_M^i} \widetilde{M}$ at $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ is the map $M \xrightarrow{X_\alpha^i} M$, the rank of which is independent of $\bar{\alpha}$. By Lemma 4.1, this shows that the coherent sheaves $\text{Coker } \theta_M^i$ and $\text{Im } \theta_M^i$ are locally free.

For the statements regarding fibres, note that because the tensor product is right exact, the exact sequence

$$\widetilde{M}(-i) \xrightarrow{\theta_M^i} \widetilde{M} \longrightarrow \text{Coker } \theta_M^i \longrightarrow 0$$

gives rise to an exact sequence of vector spaces

$$M \xrightarrow{X_\alpha^i} M \longrightarrow \mathrm{Coker} \theta_M^i \otimes k(\bar{\alpha}) \longrightarrow 0.$$

We therefore identify the fibre of $\mathrm{Coker} \theta_M^i$ at $\bar{\alpha}$ with $\mathrm{Coker}(X_\alpha^i, M)$. Taking the fibre of

$$0 \longrightarrow \mathrm{Im} \theta_M^i \longrightarrow \widetilde{M} \longrightarrow \mathrm{Coker} \theta_M^i \longrightarrow 0$$

at $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ and placing it in the top row of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Im} \theta_M^i \otimes k(\bar{\alpha}) & \longrightarrow & M & \longrightarrow & \mathrm{Coker} \theta_M^i \otimes k(\bar{\alpha}) \longrightarrow 0 \\ & & & & \parallel & & \parallel \\ 0 & \longrightarrow & \mathrm{Im}(X_\alpha^i, M) & \longrightarrow & M & \longrightarrow & \mathrm{Coker}(X_\alpha^i, M) \longrightarrow 0 \end{array}$$

then allows us to identify the fibre of $\mathrm{Im} \theta_M^i$ at $\bar{\alpha}$ with $\mathrm{Im}(X_\alpha^i, M)$. \square

Lemma 4.4. *If the ranks of X_α^i and X_α^{i+1} acting on M are both independent of the choice of $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ (so that $\mathrm{Im} \theta_M^i$ and $\mathrm{Im} \theta_M^{i+1}$ are locally free), then $\mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^i$ is also locally free, and the fibre of the short exact sequence*

$$0 \longrightarrow \mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^i \longrightarrow \mathrm{Im} \theta_M^i \xrightarrow{\theta_M} \mathrm{Im} \theta_M^{i+1}(1) \longrightarrow 0$$

at $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ is the short exact sequence of vector spaces

$$0 \longrightarrow \mathrm{Ker}(X_\alpha, M) \cap \mathrm{Im}(X_\alpha^i, M) \longrightarrow \mathrm{Im}(X_\alpha^i, M) \xrightarrow{X_\alpha} \mathrm{Im}(X_\alpha^{i+1}, M) \longrightarrow 0.$$

Proof. Consider the diagram of short exact sequences of locally free sheaves

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Im} \theta_M^i & \longrightarrow & \widetilde{M} & \longrightarrow & \mathrm{Coker} \theta_M^i \longrightarrow 0 \\ & & \downarrow & & \downarrow \theta_M & & \downarrow \\ 0 & \longrightarrow & \mathrm{Im} \theta_M^{i+1}(1) & \longrightarrow & \widetilde{M}(1) & \longrightarrow & \mathrm{Coker} \theta_M^{i+1}(1) \longrightarrow 0 \end{array}$$

whose fibre at $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Im}(X_\alpha^i, M) & \longrightarrow & M & \longrightarrow & \mathrm{Coker}(X_\alpha^i, M) \longrightarrow 0 \\ & & \downarrow & & \downarrow X_\alpha & & \downarrow \\ 0 & \longrightarrow & \mathrm{Im}(X_\alpha^{i+1}, M) & \longrightarrow & M & \longrightarrow & \mathrm{Coker}(X_\alpha^{i+1}, M) \longrightarrow 0. \end{array}$$

This immediately shows that the fibre of $\theta_M: \mathrm{Im} \theta_M^i \rightarrow \mathrm{Im} \theta_M^{i+1}(1)$ at $\bar{\alpha}$ is the induced map $X_\alpha: \mathrm{Im}(X_\alpha^i, M) \rightarrow \mathrm{Im}(X_\alpha^{i+1}, M)$. The proof now follows as described in Proposition 2.1 of [4]. \square

Now let M be a kE -module and N any submodule of M . The inclusion $N \subseteq M$ induces an inclusion of locally free sheaves $\tilde{N} \subseteq \tilde{M}$, and one readily confirms that the diagrams

$$\begin{array}{ccc} \tilde{N}(n) & \xrightarrow{\theta_N} & \tilde{N}(n+1) \\ \downarrow & & \downarrow \\ \tilde{M}(n) & \xrightarrow{\theta_M} & \tilde{M}(n+1) \end{array}$$

commute. It follows that $\text{Im } \theta_N^i \subseteq \text{Im } \theta_M^i$ and $\text{Ker } \theta_N^i \subseteq \text{Ker } \theta_M^i$ for each $i \geq 0$.

The following proposition will be the essential step in showing that $\mathcal{F}_i(N) = \mathcal{F}_i(M)$ in certain cases.

Proposition 4.5. *If M and N both have constant i - and $(i+1)$ -rank, then the fibre of the natural inclusion $\text{Ker } \theta_N \cap \text{Im } \theta_N^i \subseteq \text{Ker } \theta_M \cap \text{Im } \theta_M^i$ at a point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ is the inclusion of vector spaces $\text{Ker}(X_\alpha, N) \cap \text{Im}(X_\alpha^i, N) \subseteq \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^i, M)$.*

Proof. Since both M and N have constant i -rank, Lemma 4.3 tells us that the fibre of the diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im } \theta_N^i & \longrightarrow & \tilde{N} & \longrightarrow & \text{Coker } \theta_N^i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Im } \theta_M^i & \longrightarrow & \tilde{M} & \longrightarrow & \text{Coker } \theta_M^i \longrightarrow 0 \end{array}$$

at a point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im}(X_\alpha^i, N) & \longrightarrow & N & \longrightarrow & \text{Coker}(X_\alpha^i, N) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Im}(X_\alpha^i, M) & \longrightarrow & M & \longrightarrow & \text{Coker}(X_\alpha^i, M) \longrightarrow 0, \end{array}$$

where the middle map is the inclusion $N \subseteq M$. This implies that the fibre of the inclusion $\text{Im } \theta_N^i \subseteq \text{Im } \theta_M^i$ at $\bar{\alpha}$ is the natural inclusion $\text{Im}(X_\alpha^i, N) \subseteq \text{Im}(X_\alpha^i, M)$. The same argument also shows that the fibre of $\text{Im } \theta_N^{i+1} \subseteq \text{Im } \theta_M^{i+1}$ is $\text{Im}(X_\alpha^{i+1}, N) \subseteq \text{Im}(X_\alpha^{i+1}, M)$.

Now consider the diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker } \theta_N \cap \text{Im } \theta_N^i & \longrightarrow & \text{Im } \theta_N^i & \xrightarrow{\theta_N} & \text{Im } \theta_N^{i+1}(1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker } \theta_M \cap \text{Im } \theta_M^i & \longrightarrow & \text{Im } \theta_M^i & \xrightarrow{\theta_M} & \text{Im } \theta_M^{i+1}(1) \longrightarrow 0 \end{array}$$

whose terms are all locally free, and whose fibre at a point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(X_\alpha, N) \cap \text{Im}(X_\alpha^i, N) & \longrightarrow & \text{Im}(X_\alpha^i, N) & \xrightarrow{X_\alpha} & \text{Im}(X_\alpha^{i+1}, N) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^i, M) & \longrightarrow & \text{Im}(X_\alpha^i, M) & \xrightarrow{X_\alpha} & \text{Im}(X_\alpha^{i+1}, M) \longrightarrow 0. \end{array}$$

In light of the above remarks, the right two maps are the natural inclusions $\text{Im}(X_\alpha^i, N) \subseteq \text{Im}(X_\alpha^i, M)$ and $\text{Im}(X_\alpha^{i+1}, N) \subseteq \text{Im}(X_\alpha^{i+1}, M)$, respectively. This forces the map on the left to be the inclusion $\text{Ker}(X_\alpha, N) \cap \text{Im}(X_\alpha^i, N) \subseteq \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^i, M)$. \square

5. THE EQUAL IMAGES PROPERTY AND VECTOR BUNDLES

In this section we recall the notion of the equal images property for kE -modules. For such a module M , we introduce an inductive procedure for computing the vector bundles $\mathcal{F}_i(M)$. As was the case in Section 3, there is a close relationship between the structure of such modules and the geometry of $\mathbb{P}^{r-1}(k)$. Later in the section, we use this procedure to compute the vector bundles for so called ‘ W -modules’ in the case $r = 2$. The following definition first appeared in [6].

Definition 5.1. A kE -module M has the *equal images property* if the image of X_α acting on M is independent of the choice of $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$.

A useful characterisation of the equal images property is the following, which appeared as Proposition 2.5 of [6].

Proposition 5.2. A kE -module M has the equal images property if and only if the image of X_α acting on M is equal to $\text{Rad}(M)$ for all $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$.

We remark that the equal images property is rather strong. In particular, if M has the equal images property, then M has constant Jordan type, although the converse does not necessarily hold. (See Proposition 2.8 of [6].) We provide the following brief summary of the salient points in Section 2 of [6].

Proposition 5.3. The class of kE -modules with the equal images property is closed under taking direct sums, quotients and radicals.

The following is our main result regarding the inductive nature of the functors \mathcal{F}_i evaluated at modules having the equal images property.

Theorem 5.4. If M is a kE -module with the equal images property, then for all $0 \leq j < i \leq p$ we have $\mathcal{F}_i(M) \cong \mathcal{F}_{i-j}(\text{Rad}^j(M))$.

Proof. By the construction of θ_M , the map $\theta_M^j: \widetilde{M}(-i) \rightarrow \widetilde{M}(-i+j)$ factors through the vector bundle $\widetilde{\text{Rad}^j(M)}(-i+j)$. Let $\widehat{\theta}_M^j$ denote the induced map

$$\widetilde{M}(-i) \longrightarrow \widetilde{\text{Rad}^j(M)}(-i+j).$$

We then have a commutative diagram

$$\begin{array}{ccccc}
 \widetilde{M}(-i) & \xrightarrow{\widehat{\theta}_M^j} & \widetilde{\text{Rad}^j(M)}(-i+j) & \xrightarrow{\theta_{\text{Rad}^j(M)}^{i-j}} & \widetilde{\text{Rad}^j(M)} \\
 \parallel & & \downarrow & & \downarrow \\
 \widetilde{M}(-i) & \xrightarrow{\theta_M^j} & \widetilde{M}(-i+j) & \xrightarrow{\theta_M^{i-j}} & \widetilde{M}
 \end{array}$$

where the right two vertical arrows are those induced by the inclusion $\text{Rad}^j(M) \subseteq M$. We claim that $\widehat{\theta}_M^j$ is surjective, from which it will follow that the image of $\theta_{\text{Rad}^j(M)}^{i-j}$ equals that of θ_M^i . To see this, note that because θ_M^j is a map of vector bundles and the middle vertical arrow is an injection, $\widehat{\theta}_M^j$ is also a map of vector bundles. This also allows one to conclude that the fibre of $\widehat{\theta}_M^j$ at a point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ is the linear map $X_\alpha^j: M \rightarrow \text{Rad}^j(M)$. Since M has the equal images property, Propositions 5.2 and 5.3 imply that this map is always surjective. In other words, $\widehat{\theta}_M^j$ is a map of vector bundles that is surjective on fibres, hence surjective.

We next claim that there is an equality

$$\text{Ker } \theta_{\text{Rad}^j(M)} \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j} = \text{Ker } \theta_M \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j}$$

as subsheaves of \widetilde{M} . There is an obvious rightwards inclusion induced by the inclusion of modules $\text{Rad}^j(M) \subseteq M$. The reverse containment follows from the fact that the left hand side is the kernel of $\theta_{\text{Rad}^j(M)}: \text{Im } \theta_{\text{Rad}^j(M)}^{i-j} \rightarrow \text{Im } \theta_{\text{Rad}^j(M)}^{i-j+1}$ (1) which, after precomposing with the inclusion $\text{Ker } \theta_M \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j} \hookrightarrow \text{Im } \theta_{\text{Rad}^j(M)}^{i-j}$, yields the zero map.

Putting this all together, we therefore have

$$\begin{aligned}
 \mathcal{F}_i(M) &= \frac{\text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}}{\text{Ker } \theta_M \cap \text{Im } \theta_M^i} \cong \frac{\text{Ker } \theta_M \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j-1}}{\text{Ker } \theta_M \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j}} = \frac{\text{Ker } \theta_{\text{Rad}^j(M)} \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j-1}}{\text{Ker } \theta_{\text{Rad}^j(M)} \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j}} \\
 &= \mathcal{F}_{i-j}(\text{Rad}^j(M))
 \end{aligned}$$

as required. \square

Before giving an application of Theorem 5.4, we recall the theory of Chern classes and how they interact with the functors \mathcal{F}_i . Note that the Chow ring $A^*(\mathbb{P}^{r-1}(k))$ of projective space is isomorphic to the truncated polynomial ring $\mathbb{Z}[h]/h^r$. If \mathcal{F} is a vector bundle on $\mathbb{P}^{r-1}(k)$, then the *Chern class* of \mathcal{F} is the well defined polynomial class

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F})h + \cdots + c_{r-1}(\mathcal{F})h^{r-1} \in A^*(\mathbb{P}^{r-1}(k))$$

characterised by the following properties. (See Chapters 3 and 4 of [9] for details.)

- (1) $c_i(\mathcal{F}) = 0$ for all $i \geq \text{rank}(\mathcal{F})$.
- (2) If $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is a short exact sequence of vector bundles on $\mathbb{P}^{r-1}(k)$, then $c(\mathcal{F}_2) = c(\mathcal{F}_1)c(\mathcal{F}_3)$.
- (3) $c(\mathcal{O}(n)) = 1 + nh$ for all $n \in \mathbb{Z}$.

The integers $c_i(\mathcal{F})$ are called the *Chern numbers* of \mathcal{F} . We record the formula for Chern numbers of twists, which follows from Example 3.2.2 of [9].

Lemma 5.5. *If \mathcal{F} is a vector bundle on $\mathbb{P}^{r-1}(k)$, then for all $n \in \mathbb{Z}$, the i th Chern number of $\mathcal{F}(n)$ is given by*

$$c_i(\mathcal{F}(n)) = \sum_{j=0}^i n^j \binom{\text{rank}(\mathcal{F}) - i + j}{j} c_{i-j}(\mathcal{F}).$$

We shall combine this fact with the following result, which follows from Lemmas 2.2 and 2.3 of [4].

Proposition 5.6. *If M is a finitely generated kE -module, then \widetilde{M} has a filtration whose filtered quotients are $\mathcal{F}_i(M)(j)$ for all $0 \leq j < i \leq p$.*

6. APPLICATION: VECTOR BUNDLES FOR W -MODULES

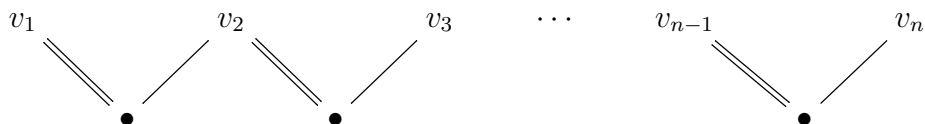
In this section we restrict our attention to the case in which E has rank two and look at W -modules for kE . Such modules were first introduced in [6] and shown there to play an important role in the theory of modules having the equal images property. Our goal here is to use the results of the previous section to compute the vector bundle $\mathcal{F}_i(M)$ for any W -module M . Again, we emphasise that throughout the section we shall require the rank E to equal two, that is, $E \cong \mathbb{Z}/p \times \mathbb{Z}/p$.

Definition 6.1. Let n and d be positive integers such that $1 \leq d \leq n$ and $d \leq p$. If V is the free kE -module of rank n with generators v_1, \dots, v_n , we define $W_{n,d}$ to be the quotient V/U , where U is the kE -submodule of V generated by the elements

$$X_1 v_1, \quad X_2 v_n, \quad X_1^d v_i \text{ for } 1 \leq i \leq n, \quad X_2 v_i - X_1 v_{i+1} \text{ for } 1 \leq i \leq n-1.$$

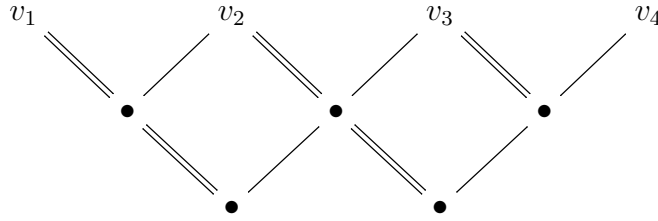
Any kE -module of the form $W_{n,d}$ is called a *W -module*.

It is convenient to picture the structure of a W -module by way of certain diagrams. For example, if p is any prime number and $n \geq 2$, then the module $W_{n,2}$ can be represented by the diagram



where each vertex represents a basis element of $W_{n,2}$. The generator X_1 of $\text{Rad}(kE)$ maps a vertex to the one lying below it, on the opposite end of a single edge. Similarly, X_2 maps a vertex to the one lying below it, on the opposite end of a double edge. We note that the Loewy length of a given module is indicated by the number of rows in the corresponding

diagram. As another example, for $p \geq 3$, the module $W_{4,3}$ has diagram



Observe that both of the above diagrams roughly have a ‘W’ shape, hence the terminology ‘W-module’.

The following appeared as Proposition 3.3 of [6].

Proposition 6.2. *If $1 \leq d \leq n$ and $d \leq p$, then the kE -module $W_{n,d}$ has the equal images property.*

As an immediate corollary, one obtains the following, which also appeared in [6].

Corollary 6.3. *If $1 \leq d \leq n$ and $d \leq p$, then $W_{n,d}$ has constant Jordan type*

$$[d]^{n-d+1}[d-1] \dots [1].$$

Proof. The fact that $W_{n,d}$ has constant Jordan type follows from Proposition 6.2. Calculating its Jordan type is then accomplished by calculating the Jordan type of X_1 on $W_{n,d}$, using the corresponding module diagram. \square

The surprising fact about W-modules is not that they have the equal images property, but that they are, in some sense, nice models for all kE -modules having the equal images property. This is made precise by the next result, which appeared as Theorem 5.4 of [6].

Proposition 6.4. *If M is a kE -module having the equal images property of radical length d , then there exists an integer $n \geq d$ and a surjective module homomorphism $W_{n,d} \rightarrow M$.*

Motivated by the central role W-modules play in the theory of kE -modules having the equal images property, we now calculate the vector bundle $\mathcal{F}_i(W_{n,d})$ for each $1 \leq i \leq d$. Before doing so, we should point out that the kernel bundle $\text{Ker } \theta_{W_{n,d}}$ was calculated in Proposition 6.4 of [6]. Given that the vector bundles $\mathcal{F}_i(W_{n,d})(j)$ form a filtration of the kernel bundle, our calculation may be viewed as a refinement of this earlier work.

We recall that Grothendieck [10] has classified the vector bundles on $\mathbb{P}^1(k)$. In particular, every such bundle is a direct sum of line bundles

$$\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(n_t),$$

where the integers n_1, \dots, n_t are uniquely determined up to reordering. Given this classification, we now present the main theorem of this section.

Theorem 6.5. *If $1 \leq d \leq n$ and $d \leq p$, then $\mathcal{F}_i(W_{n,d}) \cong \mathcal{O}_{\mathbb{P}^1}(-n+i)$ for $1 \leq i \leq d-1$, and $\mathcal{F}_d(W_{n,d}) \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d+1)}$.*

Remark 6.6. The ranks of these vector bundles are given by the exponents in the statement of Corollary 6.3.

Proof of Theorem 6.5. We proceed by induction on d , the case $d = 1$ being trivial. So suppose $d > 1$. Since the trivial bundle $\widetilde{W}_{n,d}$ has a filtration with filtered quotients $\mathcal{F}_i(M)(j)$ for $0 \leq j < i \leq d$ by Proposition 5.6, we have

$$1 = c(\widetilde{W}_{n,d}) = \prod_{0 \leq j < i \leq d} c(\mathcal{F}_i(W_{n,d})(j)).$$

Comparing the first Chern numbers using Lemma 5.5 gives us

$$(1) \quad 0 = \sum_{i=1}^d (ic_1(\mathcal{F}_i(W_{n,d})) + \frac{1}{2}i(i-1)).$$

Note that $\text{Rad}(W_{n,d})$ is also a W -module, being isomorphic to $W_{n-1,d-1}$. By induction and Theorem 5.4 we therefore have

$$\mathcal{F}_i(W_{n,d}) \cong \mathcal{F}_{i-1}(\text{Rad}(W_{n,d})) \cong \mathcal{F}_{i-1}(W_{n-1,d-1}) \cong \mathcal{O}_{\mathbb{P}^1}(-n+i)$$

for $2 \leq i \leq d-1$ and

$$\mathcal{F}_d(W_{n,d}) \cong \mathcal{F}_{d-1}(\text{Rad}(W_{n,d})) \cong \mathcal{F}_{d-1}(W_{n-1,d-1}) \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d+1)}.$$

Substituting this into (1) and simplifying yields

$$0 = c_1(\mathcal{F}_1(W_{n,d})) + n - 1$$

so that $\mathcal{F}_1(W_{n,d}) \cong \mathcal{O}_{\mathbb{P}^1}(-n+1)$. This completes the proof. \square

As a corollary, we obtain the following bit of folklore regarding the functor

$$\mathcal{F}_1: \text{cJt}(kE) \longrightarrow \text{vec}(\mathbb{P}^1(k)).$$

The fact that \mathcal{F}_1 is essentially surjective certainly follows from Theorem 1.1 of [4], but the above calculation allows us to further deduce that every vector bundle on $\mathbb{P}^1(k)$ is of the form $\mathcal{F}_1(M)$ where M has Loewy length at most two. This result was related to the first author by Dave Benson whilst the former was a student of the latter. We require a quick lemma relating vector bundles to k -linear duals of modules, which appeared as Theorem 3.6 of [4].

Lemma 6.7. *If M is a kE -module in any rank r , then $\mathcal{F}_i(M^\#) \cong \mathcal{F}_i(M)^\vee(-i+1)$.*

We are now in a position to prove the folklore indicated above.

Corollary 6.8. *Every line bundle on $\mathbb{P}^1(k)$ is isomorphic to $\mathcal{F}_1(k)$, $\mathcal{F}_1(W_{n,2})$ or $\mathcal{F}_1(W_{n,2}^\#)$ for some n .*

Proof. Theorem 6.5 tells us that $\mathcal{F}_1(k) \cong \mathcal{F}_1(W_{1,1}) \cong \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{F}_1(W_{n,2}) \cong \mathcal{O}_{\mathbb{P}^1}(-n+1)$ for all $n \geq 2$. To realise line bundles with positive Chern numbers, we use Lemma 6.7 to obtain

$$\mathcal{F}_1(W_{n,2}^\#) \cong \mathcal{F}_1(W_{n,2})^\vee \cong \mathcal{O}_{\mathbb{P}^1}(-n+1)^\vee \cong \mathcal{O}_{\mathbb{P}^1}(n-1) \quad \text{for all } n \geq 2. \quad \square$$

7. RECOLLECTIONS ABOUT THE GENERIC KERNEL FILTRATION

The category of kE -modules of constant Jordan type is wild, even in the case $r = 2$. It was shown by Benson that even the category of such modules having Loewy length three is wild. (See Section 4.5 of [3] for details.) On the other hand, the vector bundles on $\mathbb{P}^1(k)$ are rather well behaved, which leaves one to wonder whether or not there is some sort of structural invariant of a kE -module M that completely determines $\mathcal{F}_i(M)$, preferably one that is easy to understand.

For $r = 2$, it turns out that there exists a filtration of M that does allow us to compute $\mathcal{F}_i(M)$ on certain, generally much smaller subquotients of M . This filtration is related to the generic kernel of a $k(\mathbb{Z}/p)^2$ -module, which was introduced by Carlson, Friedlander and Suslin [6]. The following definition applies to kE -modules in arbitrary rank r .

Definition 7.1. Let $j \in \mathbb{N}$. A kE -module M has *constant j -rank* if the rank of X_α^j acting on M is independent of $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$. If M has constant 1-rank, then we simply say that M has *constant rank*.

Remark 7.2. It is easy to see that a kE -module has constant Jordan type if and only if it has constant j -rank for all $1 \leq j \leq p$.

Throughout the remainder of this section, we again restrict our attention to the case in which $r = 2$.

Definition 7.3. Let M be a kE -module. For any cofinite subset S of $\mathbb{P}^1(k)$, consider the submodule

$${}_S M = \sum_{\bar{\alpha} \in S} \text{Ker}(X_\alpha, M)$$

of M . The *generic kernel* of M is then defined to be the submodule

$$\mathfrak{K}(M) = \bigcap_{S \subseteq \mathbb{P}^1(k) \text{ cofinite}} {}_S M.$$

The following are the main results in Section 7 of [6] concerning the generic kernel.

Lemma 7.4. *Let M be a kE -module.*

- (1) *The generic kernel $\mathfrak{K}(M)$ has the equal images property. Moreover, if N is any submodule of M having the equal images property, then $\mathfrak{K}(M)$ contains N .*
- (2) *If M has constant rank and $\bar{\alpha} \in \mathbb{P}^1(k)$, then $\mathfrak{K}(M)$ contains the kernel of the action of X_α on M .*

Now consider the filtration

$$0 = J^p \mathfrak{K}(M) \subseteq \dots \subseteq J \mathfrak{K}(M) \subseteq \mathfrak{K}(M) \subseteq J^{-1} \mathfrak{K}(M) \subseteq \dots \subseteq J^{-p+1} \mathfrak{K}(M) = M$$

of M . (For $j \in \mathbb{N}$, $J^{-j} \mathfrak{K}(M)$ denotes the set of elements $m \in M$ for which $J^j m \subseteq \mathfrak{K}(M)$.) We call the above filtration the *generic kernel filtration* of M . Our goal is to show that the functors \mathcal{F}_i behave well with respect to the generic kernel filtration in the sense that $\mathcal{F}_i(M)$ may be computed on the subquotient $J^{-i} \mathfrak{K}(M) / J^{i+1} \mathfrak{K}(M)$. The following lemma appeared as Proposition 2.7 of [1].

Lemma 7.5. *If M is a kE -module of constant rank and $\bar{\alpha} \in \mathbb{P}^1(k)$, then for all $j \geq 0$ we have $X_\alpha^{-j} \mathfrak{K}(M) = J^{-j} \mathfrak{K}(M)$.*

An easy consequence of this is the following, which appeared as Lemma 5.24 of [2].

Lemma 7.6. *If M is a kE -module of constant rank, then for all $\bar{\alpha} \in \mathbb{P}^1(k)$ and all $i \leq j$ we have*

$$\text{Ker}(X_\alpha, J^{-j} \mathfrak{K}(M)) \cap \text{Im}(X_\alpha^i, J^{-j} \mathfrak{K}(M)) = \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^i, M).$$

Proof. The rightwards containment is clear, so let $m \in \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^i, M)$. Observe that $\text{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$ by Lemma 7.4 (2). It follows that

$$\text{Ker}(X_\alpha, J^{-j} \mathfrak{K}(M)) \subseteq \text{Ker}(X_\alpha, M) \subseteq \text{Ker}(X_\alpha, \mathfrak{K}(M)) \subseteq \text{Ker}(X_\alpha, J^{-j} \mathfrak{K}(M)),$$

whence equality holds throughout. In particular, we have $m \in \text{Ker}(X_\alpha, J^{-j} \mathfrak{K}(M))$. Also, there exists $m' \in M$ such that $X_\alpha^i m' = m$. Since $m \in \mathfrak{K}(M)$, Lemma 7.5 implies that $m' \in X_\alpha^{-i} \mathfrak{K}(M) = J^{-i} \mathfrak{K}(M) \subseteq J^{-j} \mathfrak{K}(M)$, thus $m \in \text{Im}(X_\alpha^i, J^{-j} \mathfrak{K}(M))$. \square

Although one may prove the results of the following section directly, the presentation is made considerably more elegant via the following duality statement related to the generic kernel filtration. It appeared as Theorem 3.3 of [1].

Lemma 7.7. *If M is a kE -module of constant rank and $a, b \in \mathbb{Z}$ satisfy $a \leq b$, then*

$$J^a \mathfrak{K}(M^\#) / J^b \mathfrak{K}(M^\#) \cong (J^{-b+1} \mathfrak{K}(M) / J^{-a+1} \mathfrak{K}(M))^\#.$$

8. COMPUTING $\mathcal{F}_i(M)$ IN RANK TWO VIA THE GENERIC KERNEL FILTRATION

The main theorem of this section shows that $\mathcal{F}_i(M)$ can be computed on a subquotient of M whose Loewy length is bounded in terms of i . Again, we continue to require that the rank r of E is equal to two so that $E \cong \mathbb{Z}/p \times \mathbb{Z}/p$.

The following result will allow us to employ Lemma 7.6 as the key in proving our main theorem.

Lemma 8.1. *Let $1 \leq j \leq p$ and $i \leq j + 1$. If M is a kE -module having both constant rank and constant i -rank, then the submodule $J^{-j} \mathfrak{K}(M)$ also has constant i -rank.*

Proof. The statement is clear if $i = 0$, so assume that $i \geq 1$. Let $\bar{\alpha} \in \mathbb{P}^1(k)$ and observe that if $m \in \text{Ker}(X_\alpha^i, M)$, then $X_\alpha^{i-1} m \in \text{Ker}(X_\alpha, M)$. One then has $X_\alpha^{i-1} m \in \mathfrak{K}(M)$ by Lemma 7.4 (2) so that $m \in J^{-(i-1)} \mathfrak{K}(M)$ by Lemma 7.5. Combining this with the fact that $J^{-(i-1)} \mathfrak{K}(M) \subseteq J^{-j} \mathfrak{K}(M)$ shows that

$$\text{Ker}(X_\alpha^i, J^{-j} \mathfrak{K}(M)) = \text{Ker}(X_\alpha^i, M).$$

Because M has constant i -rank, the dimension of the right hand term is independent of $\bar{\alpha}$, hence so is the dimension of the left hand term. \square

Proposition 8.2. *If M is a kE -module of constant Jordan type and $i \leq j$, then*

$$\mathcal{F}_i(M) \cong \mathcal{F}_i(J^{-j} \mathfrak{K}(M)).$$

Proof. Let $N = J^{-j}\mathfrak{K}(M)$. The inclusion $N \subseteq M$ induces an inclusion of coherent sheaves $\widetilde{N} \subseteq \widetilde{M}$, which in turn yields a natural inclusion

$$(2) \quad \text{Ker } \theta_N \cap \text{Im } \theta_N^{i-1} \subseteq \text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}.$$

By Proposition 8.1, since M has constant Jordan type, N has constant $(i-1)$ -rank and constant i -rank. Using Lemma 4.5, this implies that the fibre of the inclusion (2) at any point $\bar{\alpha} \in \mathbb{P}^1(k)$ is just the inclusion of vector spaces

$$\text{Ker}(X_\alpha, N) \cap \text{Im}(X_\alpha^{i-1}, N) \subseteq \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^{i-1}, M).$$

But the latter inclusion is an equality for all $\bar{\alpha}$ by Lemma 7.6, so we actually have

$$\text{Ker } \theta_N \cap \text{Im } \theta_N^{i-1} = \text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}.$$

An identical argument also shows that $\text{Ker } \theta_N \cap \text{Im } \theta_N^i = \text{Ker } \theta_M \cap \text{Im } \theta_M^i$, hence

$$\mathcal{F}_i(M) = \frac{\text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}}{\text{Ker } \theta_M \cap \text{Im } \theta_M^i} = \frac{\text{Ker } \theta_N \cap \text{Im } \theta_N^{i-1}}{\text{Ker } \theta_N \cap \text{Im } \theta_N^i} = \mathcal{F}_i(N). \quad \square$$

Theorem 8.3. *If M is a kE -module of constant Jordan type and $i \leq \min\{j, \ell - 1\}$, then*

$$\mathcal{F}_i(M) \cong \mathcal{F}_i(J^{-j}\mathfrak{K}(M)/J^\ell\mathfrak{K}(M)).$$

Proof. We have $\mathcal{F}_i(M) = \mathcal{F}_i(J^{-j}\mathfrak{K}(M))$ by Proposition 8.2. Lemma 6.7 then tells us that $\mathcal{F}_i(M^\#) \cong \mathcal{F}_i((J^{-j}\mathfrak{K}(M))^\#)$. Note by Lemma 7.7 that we have

$$(J^{-j}\mathfrak{K}(M))^\# \cong M^\# / J^{j+1}\mathfrak{K}(M^\#),$$

where the former module has constant $(i-1)$ -rank, i -rank and $(i+1)$ -rank, hence the latter does as well. Proposition 8.2 then shows that

$$\mathcal{F}_i(M^\# / J^{j+1}\mathfrak{K}(M^\#)) = \mathcal{F}_i(J^{-\ell}\mathfrak{K}(M^\#) / J^{j+1}\mathfrak{K}(M^\#)).$$

Putting this together now yields

$$\mathcal{F}_i(M^\#) \cong \mathcal{F}_i(J^{-\ell}\mathfrak{K}(M^\#) / J^{j+1}\mathfrak{K}(M^\#)).$$

Using Lemma 7.7 then gives us

$$\mathcal{F}_i(M^\#) \cong \mathcal{F}_i((J^{-j}\mathfrak{K}(M) / J^{\ell+1}\mathfrak{K}(M))^\#),$$

and another use of Lemma 6.7 yields

$$\mathcal{F}_i(M) \cong \mathcal{F}_i(J^{-j}\mathfrak{K}(M) / J^{\ell+1}\mathfrak{K}(M))$$

as desired. □

The strongest form of Theorem 8.3 is the following.

Corollary 8.4. *If M is a kE -module of constant Jordan type, then*

$$\mathcal{F}_i(M) \cong \mathcal{F}_i(J^{-i}\mathfrak{K}(M) / J^{i+1}\mathfrak{K}(M)).$$

Example 8.5. If M is a kE -module having the equal images property, then for any $i \geq 1$, $\text{Rad}^{i-1}(M) = J^{i-1}M$ also has the equal images property, hence $\mathfrak{K}(J^{i-1}(M)) = J^{i-1}M$. It follows from Theorem 5.4 and Corollary 8.4 that

$$\mathcal{F}_i(M) \cong \mathcal{F}_1(J^{i-1}M) \cong \mathcal{F}_1(J^{i-1}M/J^{i+1}M).$$

Note that the subquotient $J^{i-1}M/J^{i+1}M$ again has the equal images property, and what's more, it has Loewy length at most two. One may verify that such modules are isomorphic to direct sums of W -modules of the form $W_{n,2}$. The techniques in Section 6 may therefore be applied in computing $\mathcal{F}_i(M)$ for any module having the equal images property.

9. THE n TH POWER GENERIC KERNEL AND HIGHER RANKS

It turns out that, although the generic kernel filtration is suitable for detecting how the functors \mathcal{F}_i behave with respect to a kE -module M , there are even smaller subquotients of M that do a better job. In the most general setting the theory even carries over to higher ranks. We begin our exposition in this broader context.

Definition 9.1. Let E be an elementary abelian p -group of arbitrary rank r , let M be a kE -module, and fix $n > 0$. For any dense open subset $U \subseteq \mathbb{P}^{r-1}(k)$, let

$${}^n_U M = \sum_{\bar{\alpha} \in U} \text{Ker}(X_{\bar{\alpha}}^n, M).$$

The n th power generic kernel of M is defined to be the submodule

$$\mathfrak{K}^n(M) = \bigcap_{U \subseteq \mathbb{P}^{r-1}(k) \text{ dense open}} {}^n_U M$$

of M .

Remark 9.2. Our definition of the n th power generic kernel is a trivial extension of that given in [6] for the case $r = 2$. As was the case for generic kernels in rank two, because M is finite dimensional, we know that there always exists a dense open subset $U \subseteq \mathbb{P}^{r-1}(k)$ for which $\mathfrak{K}^n(M) = {}^n_U M$. If M has constant n -rank, the next proposition shows that one may take U to be all of $\mathbb{P}^{r-1}(k)$.

Proposition 9.3. *If M is a kE -module of constant n -rank, then*

$$\mathfrak{K}^n(M) = {}^n_{\mathbb{P}^{r-1}(k)} M = \sum_{\bar{\alpha} \in \mathbb{P}^{r-1}(k)} \text{Ker}(X_{\bar{\alpha}}^n, M).$$

Proof. We borrow the technique used in Proposition 7.6 of [6].

Write $\mathfrak{K}^n(M) = {}^n_U M$ for some dense open $U \subseteq \mathbb{P}^{r-1}(k)$. By the proof of Lemma 1.2 of [8], the points $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ for which $X_{\bar{\alpha}}^n$ has maximal rank on $\mathfrak{K}^n(M)$ also form a dense open subset of $\mathbb{P}^{r-1}(k)$. These open subsets intersect non-trivially, hence there exists a point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ such that $\mathfrak{K}^n(M)$ contains $\text{Ker}(X_{\bar{\alpha}}^n, M)$ and the rank of $X_{\bar{\alpha}}^n$ on $\mathfrak{K}^n(M)$ is

maximal. For any point $\bar{\beta} \in \mathbb{P}^{r-1}(k)$ we then have

$$\begin{aligned} \dim_k \operatorname{Ker}(X_\alpha^n, \mathfrak{K}^n(M)) &\leq \dim_k \operatorname{Ker}(X_\beta^n, \mathfrak{K}^n(M)) \\ &\leq \dim_k \operatorname{Ker}(X_\beta^n, M) \\ &= \dim_k \operatorname{Ker}(X_\alpha^n, M) \\ &= \dim_k \operatorname{Ker}(X_\alpha^n, \mathfrak{K}^n(M)). \end{aligned}$$

Here the first inequality holds since X_α^n has maximal rank on $\mathfrak{K}^n(M)$, the second inequality follows from the fact that $\mathfrak{K}^n(M)$ is a submodule of M , the first equality holds because M has constant n -rank, and the second equality follows from the fact that $\operatorname{Ker}(X_\alpha^n, M)$ is contained in $\mathfrak{K}^n(M)$. In particular, this shows that $\dim_k \operatorname{Ker}(X_\beta^n, \mathfrak{K}^n(M)) = \dim_k \operatorname{Ker}(X_\beta^n, M)$, forcing us to have $\operatorname{Ker}(X_\beta^n, M) \subseteq \mathfrak{K}^n(M)$. \square

We observe that the n th power generic kernel has a ‘dual’ construction.

Definition 9.4. Let E be an elementary abelian p -group having arbitrary rank r and let M be a kE -module. We define the n th power generic image of M to be the submodule

$$\mathfrak{I}^n(M) = \bigcap_{\bar{\alpha} \in \mathbb{P}^{r-1}(k)} \operatorname{Im}(X_\alpha^n, M)$$

of M .

Using the very same proof as that given in Proposition 8.4 of [6], one readily establishes the following, which shows how the n th power generic kernel and n th power generic image are related via duality.

Proposition 9.5. For any kE -module M we have $\mathfrak{K}^n(M^\#) \cong (\mathfrak{I}^n(M))^\perp$.

10. THE n TH POWER GENERIC KERNEL AND THE EQUAL n -IMAGES PROPERTY

We take a brief side trip here to explain what the n th power generic kernel is, or rather, what it is not. The nomenclature would seem to suggest that $\mathfrak{K}^n(M)$ might be the fibre of the generic operator θ_M^n at any point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$, but we shall see that this is not the case unless M has a very strong property. We first generalise Definition 5.1 in the obvious way.

Definition 10.1. A kE -module M has the *equal n -images property* if the image of X_α^n acting on M is independent of the choice of $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$.

We now give a general lemma.

Lemma 10.2. Let X be a variety and \mathcal{E} a vector bundle on X . Let \mathcal{E}_1 and \mathcal{E}_2 be subbundles of \mathcal{E} such that $\mathcal{E}/\mathcal{E}_1$ and $\mathcal{E}/\mathcal{E}_2$ are both locally free. Then \mathcal{E}_1 is a subbundle of \mathcal{E}_2 inside \mathcal{E} if and only if $\mathcal{E}_1 \otimes k(x) \subseteq \mathcal{E}_2 \otimes k(x)$ for each closed point $x \in X$.

Proof. Suppose first that $\mathcal{E}_1 \subseteq \mathcal{E}_2$. By the universal property of cokernels, there is a unique morphism $\mathcal{E}/\mathcal{E}_1 \rightarrow \mathcal{E}/\mathcal{E}_2$ making the following diagram commute.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/\mathcal{E}_1 \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/\mathcal{E}_2 \longrightarrow 0 \end{array}$$

Since all sheaves are locally free, both rows remain exact after tensoring with $k(x)$ for any $x \in X$. The fibre of the left vertical arrow is therefore an inclusion $\mathcal{E}_1 \otimes k(x) \subseteq \mathcal{E}_2 \otimes k(x)$. Conversely, suppose that $\mathcal{E}_1 \otimes k(x) \subseteq \mathcal{E}_2 \otimes k(x)$ for all $x \in X$ and let π be the composition $\mathcal{E}_1 \hookrightarrow \mathcal{E} \rightarrow \mathcal{E}/\mathcal{E}_2$. After base changing to $k(x)$, we have an exact sequence

$$\mathrm{Ker} \pi \otimes k(x) \longrightarrow \mathcal{E}_1 \otimes k(x) \longrightarrow \mathrm{Im} \pi \otimes k(x) \longrightarrow 0.$$

The map $\pi \otimes k(x)$ is the composition

$$\mathcal{E}_1 \otimes k(x) \longrightarrow \mathrm{Im} \pi \otimes k(x) \hookrightarrow (\mathcal{E}/\mathcal{E}_2) \otimes k(x) \cong \frac{\mathcal{E} \otimes k(x)}{\mathcal{E}_2 \otimes k(x)}.$$

By our assumption, the first map is zero for all $x \in X$, hence $\mathrm{Im} \pi \otimes k(x) = 0$ for all closed points $x \in X$. Since $\mathrm{Im} \pi$ is coherent, we have $\mathrm{Im} \pi = 0$ so that $\mathcal{E}_1 \subseteq \mathcal{E}_2$. \square

Proposition 10.3. *If M is a kE -module of constant Jordan type, then $\mathrm{Ker} \theta_M^n \subseteq \widetilde{\mathfrak{K}^n(M)}$. Furthermore, equality holds if and only if M has the equal n -images property.*

Proof. First note that if M has constant Jordan type, then M has constant n -rank for all $n > 0$. It follows from Proposition 9.3 that $\mathfrak{K}^n(M) = \sum_{\bar{\alpha} \in \mathbb{P}^{r-1}(k)} \mathrm{Ker}(X_{\bar{\alpha}}^n, M)$.

It is obvious that $\mathrm{Ker} \theta_M^n$ is a subsheaf of \widetilde{M} . The inclusion $\mathfrak{K}^n(M) \subseteq M$ also identifies $\widetilde{\mathfrak{K}^n(M)}$ as a subsheaf of \widetilde{M} . Since M has constant Jordan type, $\mathrm{Im} \theta_M^n$ is locally free. The sheaf $M/\widetilde{\mathfrak{K}^n(M)}$ is also locally free, so $\mathrm{Ker} \theta_M^n$ and $\widetilde{\mathfrak{K}^n(M)}$ both satisfy the initial hypotheses of Lemma 10.2.

To show that $\mathrm{Ker} \theta_M^n \subseteq \widetilde{\mathfrak{K}^n(M)}$ it suffices, using Lemma 10.2, to prove the inclusion on fibres. But this is clear since

$$\mathrm{Ker} \theta_M^n \otimes k(\bar{\alpha}) = \mathrm{Ker}(X_{\bar{\alpha}}^n, M) \subseteq \sum_{\bar{\beta} \in \mathbb{P}^{r-1}(k)} \mathrm{Ker}(X_{\bar{\beta}}^n, M) = \mathfrak{K}^n(M).$$

This establishes the first statement.

Now suppose that $\mathrm{Ker} \theta_M^n = \widetilde{\mathfrak{K}^n(M)}$. We have

$$\mathrm{Ker}(X_{\bar{\alpha}}^n, M) = \mathrm{Ker} \theta_M^n \otimes k(\bar{\alpha}) = \widetilde{\mathfrak{K}^n(M)} = \sum_{\bar{\beta} \in \mathbb{P}^{r-1}(k)} \mathrm{Ker}(X_{\bar{\beta}}^n, M)$$

for all $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$, which shows that M has the equal n -images property.

Conversely, if M has the equal n -images property, then $\mathrm{Ker} \theta_M^n \otimes k(\bar{\alpha}) = \mathfrak{K}^n(M)$ for all $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$. The reverse implication in Lemma 10.2 then shows that $\mathrm{Ker} \theta_M^n \subseteq \widetilde{\mathfrak{K}^n(M)}$.

Since these are vector bundles of the same rank with equal fibres, we must have $\text{Ker } \theta_M^n = \widetilde{\mathfrak{K}^n(M)}$. □

11. COMPUTING $\mathcal{F}_i(M)$ USING n TH POWER GENERIC KERNELS

In this section we show that the n th power generic kernels and n th power generic images of a kE -module M can be used to compute the vector bundles $\mathcal{F}_i(M)$ for a kE -module M of constant Jordan type. Again, our discussion applies to the case where E has arbitrary rank r .

Lemma 11.1. *Let M be a kE -module and $n \geq 0$. Then for all $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ and all $j \leq n$, the j -rank of X_α on $\mathfrak{K}^n(M)$ is equal to the j -rank of X_α on M .*

Proof. Since $j \leq n$, we have $\text{Ker}(X_\alpha^j, M) \subseteq \text{Ker}(X_\alpha^n, M) \subseteq \mathfrak{K}^n(M)$. It immediately follows that $\text{Ker}(X_\alpha^j, \mathfrak{K}^n(M)) = \text{Ker}(X_\alpha^j, M)$, thus $\text{rank}(X_\alpha^j, \mathfrak{K}^n(M)) = \text{rank}(X_\alpha^j, M)$ by the rank-nullity theorem. □

Proposition 11.2. *If $j \leq n$, then $\mathfrak{K}^n(M)$ has constant j -rank if and only if M has constant j -rank.*

Proof. Immediate from the lemma. □

The following should be compared with Lemma 7.6.

Lemma 11.3. *Let M be a kE -module having constant n -rank for some $n \geq 0$. Then for all $i \leq n - 1$ and all $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ we have*

$$\text{Ker}(X_\alpha, \mathfrak{K}^n(M)) \cap \text{Im}(X_\alpha^i, \mathfrak{K}^n(M)) = \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^i, M).$$

Proof. The rightwards containment is obvious since $\mathfrak{K}^n(M)$ is a submodule of M . For the reverse containment, note that if $m \in M$ satisfies $X_\alpha m = 0$ and there exists $m' \in M$ such that $X_\alpha^i m' = m$, then $m' \in \text{Ker}(X_\alpha^{i+1}, M)$. Using the characterisation

$$\mathfrak{K}^n(M) = \sum_{\bar{\beta} \in \mathbb{P}^{r-1}(k)} \text{Ker}(X_\beta^n, M),$$

this shows that $m' \in \mathfrak{K}^n(M)$ so that $m \in \text{Im}(X_\alpha^i, \mathfrak{K}^n(M))$. □

Proposition 11.4. *If M is a kE -module having constant j -rank for all $j \leq n$, then for all $i \leq n - 1$ we have $\mathcal{F}_i(M) = \mathcal{F}_i(\mathfrak{K}^n(M))$.*

Proof. By Proposition 11.2, $\mathfrak{K}^n(M)$ has constant $(i - 1)$ -rank, i -rank and $(i + 1)$ -rank. For $j = i - 1$ and i , it follows by Proposition 4.5 that the fibres of the inclusions

$$\text{Ker } \theta_{\mathfrak{K}^n(M)} \cap \text{Im } \theta_{\mathfrak{K}^n(M)}^j \subseteq \text{Ker } \theta_M \cap \text{Im } \theta_M^j$$

at any point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ are the inclusions of vector spaces

$$\text{Ker}(X_\alpha, \mathfrak{K}^n(M)) \cap \text{Im}(X_\alpha^j, \mathfrak{K}^n(M)) \subseteq \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^j, \theta_M).$$

By Lemma 11.3, these inclusions are equalities, and because all of the above sheaves are locally free (see Lemma 4.4), we conclude that

$$\mathrm{Ker} \theta_{\mathfrak{R}^n(M)} \cap \mathrm{Im} \theta_{\mathfrak{R}^n(M)}^j = \mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^j.$$

We therefore have

$$\mathcal{F}_i(M) = \frac{\mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^{i-1}}{\mathrm{Ker} \theta_M \cap \mathrm{Im} \theta_M^i} = \frac{\mathrm{Ker} \theta_{\mathfrak{R}^n(M)} \cap \mathrm{Im} \theta_{\mathfrak{R}^n(M)}^{i-1}}{\mathrm{Ker} \theta_{\mathfrak{R}^n(M)} \cap \mathrm{Im} \theta_{\mathfrak{R}^n(M)}^i} = \mathcal{F}_i(\mathfrak{R}^n(M)). \quad \square$$

In light of the duality that exists between \mathfrak{R}^n and \mathfrak{I}^n , we now show that quotienting out by $\mathfrak{I}^n(M)$ has the same effect as taking the submodule $\mathfrak{R}^n(M)$.

Corollary 11.5. *If M is a kE -module that has constant j -rank for all $j \leq n$, then for all $i \leq n - 1$ we have $\mathcal{F}_i(M) = \mathcal{F}_i(M/\mathfrak{I}^n(M))$.*

Proof. By Lemma 6.7, Theorem 11.4 and Proposition 9.5, respectively, we compute

$$\begin{aligned} \mathcal{F}_i(M) &\cong \mathcal{F}_i(M^\#)^\vee(-i+1) = \mathcal{F}_i(\mathfrak{R}^n(M^\#))^\vee(-i+1) \cong \mathcal{F}_i(\mathfrak{I}^n(M)^\perp)^\vee(-i+1) \\ &\cong \mathcal{F}_i((M/\mathfrak{I}^n(M))^\#)^\vee(-i+1) \cong \mathcal{F}_i(M/\mathfrak{I}^n(M)). \end{aligned} \quad \square$$

Before presenting the main theorem of the section, we give a somewhat obvious lemma regarding the relationship between generic n -kernels and generic n -images.

Lemma 11.6. *Let M be a kE -module and $n \geq 0$. Then for all $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ and all $j \leq n$, the j -rank of X_α on $M/\mathfrak{I}^n(M)$ is equal to the j -rank of X_α on M .*

Proof. This follows from Lemma 11.1 and the duality formula 9.5. □

We are now ready to prove our main result.

Theorem 11.7. *If M has constant Jordan type and $i \leq \min\{n-1, m-1\}$, then*

$$\mathcal{F}_i(M) \cong \mathcal{F}_i(\mathfrak{R}^n(M)/\mathfrak{I}^m \mathfrak{R}^n(M)) \quad \text{and} \quad \mathcal{F}_i(M) \cong \mathcal{F}_i(\mathfrak{R}^n(M/\mathfrak{I}^m(M))).$$

Proof. By Lemmas 11.1 and 11.6, every submodule and every quotient in the above formulæ has constant j -rank for all $j \leq \min\{n, m\}$. The proof therefore follows by the appropriate use of Proposition 11.4 and Corollary 11.5. □

Again, we specify the above theorem in its strongest form.

Corollary 11.8. *If M has constant Jordan type, then*

$$\mathcal{F}_i(M) \cong \mathcal{F}_i(\mathfrak{R}^{i+1}(M)/\mathfrak{I}^{i+1} \mathfrak{R}^{i+1}(M)) \quad \text{and} \quad \mathcal{F}_i(M) \cong \mathcal{F}_i(\mathfrak{R}^{i+1}(M/\mathfrak{I}^{i+1}(M))).$$

We show in the following section how these results may be applied to the rank two case in better understanding how to compute the bundles $\mathcal{F}_i(M)$ by restricting to subquotients of M having smaller Loewy length.

12. SOME EXAMPLES AND APPLICATIONS

It is now time to apply the somewhat technical results of the previous sections to some elementary, concrete examples. We shall focus our attention on the case $r = 2$, which in some sense is easier to work with.

It was shown in [6] that if M has constant rank, then $\mathfrak{K}^n(M)$ is contained in $J^{-n+1}\mathfrak{K}(M)$. This means that we have a series of inclusions

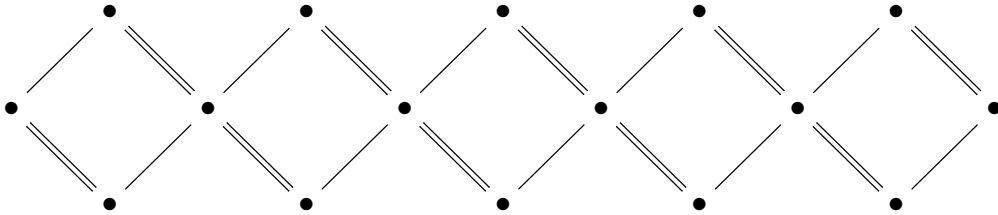
$$\begin{array}{ccccccccc} \mathfrak{K}(M) & \subseteq & J^{-1}\mathfrak{K}(M) & \subseteq & J^{-2}\mathfrak{K}(M) & \subseteq & \cdots & \subseteq & J^{-p+1}\mathfrak{K}(M) \\ \parallel & & \cup & & \cup & & & & \parallel \\ \mathfrak{K}^1(M) & \subseteq & \mathfrak{K}^2(M) & \subseteq & \mathfrak{K}^3(M) & \subseteq & \cdots & \subseteq & \mathfrak{K}^p(M) = M \end{array}$$

and dually

$$\begin{array}{ccccccccc} J\mathfrak{K}(M) & \supseteq & J^2\mathfrak{K}(M) & \supseteq & J^3\mathfrak{K}(M) & \supseteq & \cdots & \supseteq & J^p\mathfrak{K}(M) \\ \parallel & & \cap & & \cap & & & & \parallel \\ \mathfrak{I}(M) & \supseteq & \mathfrak{I}^2(M) & \supseteq & \mathfrak{I}^3(M) & \supseteq & \cdots & \supseteq & \mathfrak{I}^p(M) = 0 \end{array}$$

In general, the vertical inclusions can be strict, i.e., $\mathfrak{K}^n(M)/\mathfrak{I}^m\mathfrak{K}^n(M)$ and $\mathfrak{K}^n(M)/\mathfrak{I}^m(M)$ tend to be strictly smaller subquotients of $J^{-n+1}\mathfrak{K}(M)/J^m\mathfrak{K}(M)$.

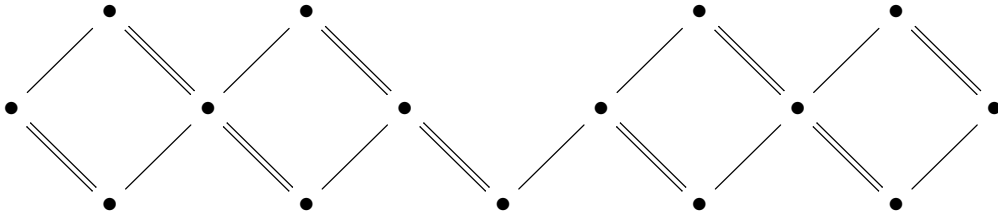
Example 12.1. Let $p = 3$ and M the kE -module given by the following diagram.



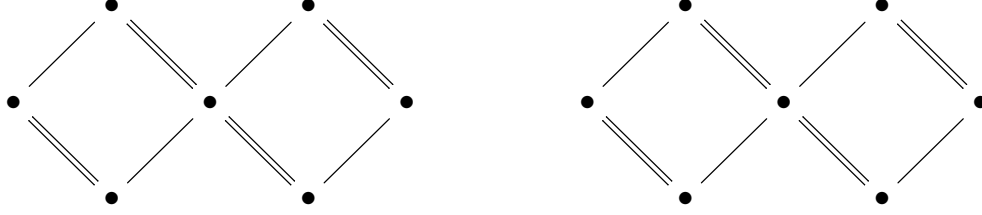
One may calculate that M is a module of constant Jordan type $[3]^4[2]^2$. Moreover,

$$M = J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M).$$

On the other hand, $\mathfrak{K}^2(M)$ is given by the diagram



and $\mathfrak{K}^2(M)/\mathfrak{J}^2\mathfrak{K}^2(M)$ has diagram



One may check that the latter diagram is also that for $\mathfrak{K}^2(M/\mathfrak{J}^2(M))$.

It is easy to check that both constructions $\mathfrak{K}^n(-)/\mathfrak{J}^m\mathfrak{K}^n(-)$ and $\mathfrak{K}^n(-/\mathfrak{J}^m(-))$ are functorial. We do not know of an example in which they are not naturally isomorphic.

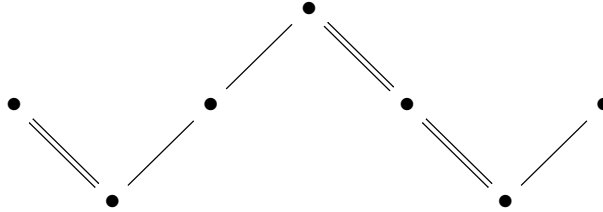
Question 12.2. If M is any kE -module, is it always the case that

$$\mathfrak{K}^n(M)/\mathfrak{J}^m\mathfrak{K}^n(M) \cong \mathfrak{K}^n(M/\mathfrak{J}^m(M))?$$

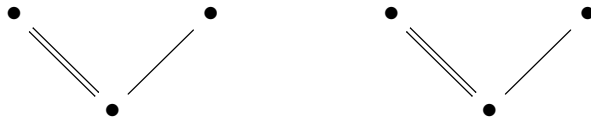
If not, then is the statement true whenever M has constant Jordan type, for example?

We now show how the results of the paper may help us compute $\mathcal{F}_1(M)$ in an extremely efficient way.

Example 12.3. Consider the seven dimensional kE -module M given by the diagram



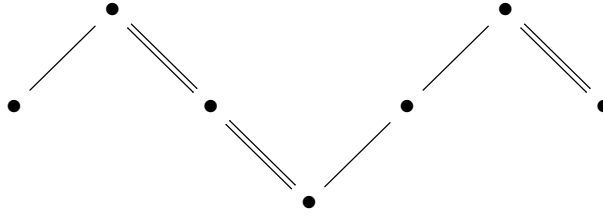
This first appeared in [6]. To show how the results of this paper make the computation of $\mathcal{F}_1(M)$ almost trivial, note that $\mathfrak{K}^2(M)$ is the submodule given by the diagram



By the discussion in Section 6, one sees that this is just two copies of the W -module $W_{2,2}$. It now follows from Proposition 11.4 and Theorem 6.5 that

$$\mathcal{F}_1(M) = \mathcal{F}_1(\mathfrak{K}^2(M)) \cong \mathcal{F}_1(W_{2,2}) \oplus \mathcal{F}_1(W_{2,2}) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Dually, consider the k -linear dual $M^\#$, whose structure is given by the dual diagram



In this case, $\mathfrak{K}^2(M/\mathfrak{J}^2(M)) \cong M/\mathfrak{J}^2(M)$, which is given by the diagram



This is now two copies of the dual W -module $W_{n,2}^\#$. From this we see that

$$\mathcal{F}_1(M^\#) = \mathcal{F}_1(M/\mathfrak{J}^2(M)) \cong \mathcal{F}_1(W_{2,2}^\#) \oplus \mathcal{F}_1(W_{2,2}^\#) \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

13. A DISCUSSION ABOUT VECTOR BUNDLES IN RANK TWO

Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$ and recall that for $p > 2$, kE has wild representation type. What's more, it was shown by Benson that the subcategory $\mathbf{cJt}(kE)$ of modules of constant Jordan type is also wild. On the other hand, we know that the functor $\mathcal{F}_1: \mathbf{cJt}(kE) \rightarrow \mathbf{vec}(\mathbb{P}^1(k))$ is essentially surjective, and the right hand side is certainly tame by Grothendieck's theorem. One of the original goals of this project was to find a structural invariant $G(M)$ of M landing in some tame subcategory of $\mathbf{mod}(kE)$ such that $\mathcal{F}_1(M) \cong \mathcal{F}_1(G(M))$. The functorial relationship between modules of constant Jordan type and vector bundles would then reduce to one between two tame categories.

It was shown in Corollary 8.4 that $\mathcal{F}_1(M) \cong \mathcal{F}_1(J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M))$, so one might hope that the modules M for which $J^{-1}\mathfrak{K}(M) = M$ and $J^2\mathfrak{K}(M) = 0$ form a tame subcategory, or at least those M of the form $J^{-1}\mathfrak{K}(N)/J^2\mathfrak{K}(N)$ for some module N of constant Jordan type. This might also appear somewhat plausible due to the fact that such modules have Loewy length only three.

Unfortunately, Benson has also shown that the category of modules M having constant Jordan type satisfying $J^{-1}\mathfrak{K}(M) = M$ and $J^2\mathfrak{K}(M) = 0$ is in fact wild. (See Section 5 of [3] for details.) Our investigation has led us to consider the functors $\mathfrak{K}^2(-)/\mathfrak{J}^2\mathfrak{K}^2(-)$ and $\mathfrak{K}^2(-/\mathfrak{J}^2(-))$ as more likely candidates. As can be seen from Example 12.3, what appears to happen is that both $\mathfrak{K}^2(M)/\mathfrak{J}^2\mathfrak{K}^2(M)$ and $\mathfrak{K}^2(M/\mathfrak{J}^2(M))$ break up into direct sums of W -modules (or duals of W -modules) having Loewy length at most two, the decomposition of which allows one to instantly calculate $\mathcal{F}_1(M)$ using Theorem 6.5. Although these subquotients might still contain direct summands of Loewy length three, we conjecture that such summands contribute nothing to $\mathcal{F}_1(M)$. We make this more precise.

Conjecture 13.1. *If M is a $k(\mathbb{Z}/p)^2$ -module of constant Jordan type and N is any indecomposable direct summand of $\mathfrak{K}^2(M)/\mathfrak{J}^2\mathfrak{K}^2(M)$ or $\mathfrak{K}^2(M/\mathfrak{J}^2(M))$ that has Loewy length three, then $\mathcal{F}_1(N)$ is the zero sheaf on $\mathbb{P}^1(k)$.*

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