

MODULES OF CONSTANT JORDAN TYPE WITH TWO NON-PROJECTIVE BLOCKS

SHAWN BALAND

ABSTRACT. Let E be an elementary abelian p -group of rank r and let k be an algebraically closed field of characteristic p . We investigate finitely generated kE -modules M of stable constant Jordan type $[a][b]$ for $1 \leq a, b \leq p-1$ using the functors \mathcal{F}_i from modules of constant Jordan type to vector bundles on projective space \mathbb{P}^{r-1} constructed by Benson and Pevtsova [3].

In particular, we study relations on the first few Chern numbers of the trivial bundle \tilde{M} to obtain restrictions on the values of a and b for sufficiently large ranks and primes. Finally, we use similar techniques to find restrictions on the values of p and r for which there exist modules of stable constant Jordan type $[3][2][1]$.

1. INTRODUCTION

Let $E = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian p -group of rank r and let k be an algebraically closed field of characteristic p . Setting $X_i = g_i - 1 \in kE$, we define for each nonzero $\alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$ the element $X_\alpha = \lambda_1 X_1 + \dots + \lambda_r X_r \in kE$. Since $X_\alpha^p = 0$, X_α acts on a finitely generated kE -module M via a matrix whose Jordan canonical form has Jordan blocks of length at most p . We say that X_α has *Jordan type* $[p]^{a_p} \dots [1]^{a_1}$ on M if X_α acts on M with a_j Jordan blocks of length j . We say that M has *constant Jordan type* $[p]^{a_p} \dots [1]^{a_1}$ if the Jordan type of X_α on M is $[p]^{a_p} \dots [1]^{a_1}$ for all $0 \neq \alpha \in \mathbb{A}^r(k)$. In this case we call $[p-1]^{a_{p-1}} \dots [1]^{a_1}$ the *stable Jordan type* of M .

Modules of constant Jordan type were introduced by Carlson, Friedlander and Pevtsova [5] in the context of finite group schemes and have been studied more recently in [4, 1, 3, 2] for elementary abelian p -groups. One of the main objectives in studying these modules is to better understand which stable Jordan types are realized as the stable Jordan types of kE -modules having constant Jordan type.

For example, Benson [1] proved there is no kE -module of stable constant Jordan type $[a]$ for $2 \leq a \leq p-2$ and $p \geq 5$. More recently, Benson and Pevtsova [3] have constructed functors \mathcal{F}_i ($1 \leq i \leq p$) from finitely generated kE -modules of constant Jordan type to vector bundles on projective space \mathbb{P}^{r-1} and investigated the Chern numbers of the vector bundles $\mathcal{F}_i(M)$. Benson [2] used the conclusions of [3] to prove the following.

Theorem 1.1 (Benson [2]). *If a kE -module has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ and $\sum_{j=1}^{p-1} j a_j \leq \min(r-1, p-2)$, then $a_j = 0$ for all $2 \leq j \leq p-1$.*

In this paper we introduce a variation of the technique used in proving Theorem 1.1 to study modules of stable constant Jordan type $[a][b]$. In particular, we use relations modulo

p on the first few Chern numbers of the trivial sheaf \tilde{M} with constant fibre M to obtain restrictions on a and b . The following are the main results of this paper.

Theorem 1.2. *Let $p \geq 5$ and $r \geq 4$. If a kE -module has stable constant Jordan type $[a][b]$ with $a \neq b$, then one of the following holds.*

- (i) $a = p - b$.
- (ii) $a = p - b \pm 1$.
- (iii) $a^2 + b^2 - ab - 1 \equiv 0 \pmod{p}$.

Theorem 1.3. *Let $p \geq 7$ and $r \geq 5$. If a kE -module has stable constant Jordan type $[a]^2$ with $2 \leq a \leq p - 2$, then $a = \frac{p-1}{2}$ or $a = \frac{p+1}{2}$.*

We should mention that these restrictions are by no means sharp, and it is believed that very few stable Jordan types $[a][b]$ are actually realized by kE -modules of constant Jordan type. In particular, a conjecture of Rickard states that if M has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ and $a_j = 0$, then the number of Jordan blocks of length greater than j is divisible by p . This statement would rule out the existence of modules having stable constant Jordan types $[a][b]$ with $2 \leq a < b \leq p - 2$, $[a][1]$ with $3 \leq a \leq p - 2$ and $[p - 1][a]$ with $2 \leq a \leq p - 3$. A conjecture of Suslin (see Conjecture 3.5) would also rule out the existence of modules having stable constant Jordan type $[a]^2$ for $2 \leq a \leq p - 2$. Another conjecture appearing in [5] states that if $p \geq 5$ and M has stable constant Jordan type $[2][1]^j$, then $j \geq r - 1$. This would rule out the existence of a module M having stable constant Jordan type $[2][1]$ for $r > 2$, and hence also a module of stable constant Jordan type $[p - 1][p - 2]$ since $\Omega(M)$ would necessarily have this stable constant Jordan type.

There are examples of modules having stable constant Jordan type $[2][1]$ for $r = 2$, however. In particular, the module $kE/J^2(kE)$ has this stable constant Jordan type for any prime p . The remaining stable Jordan types $[a][b]$ not excluded by the above conjectures, namely $[1]^2$, $[p - 1]^2$ and $[p - 1][1]$, are indeed realized in any rank by the modules $k \oplus k$, $\Omega(k) \oplus \Omega(k)$ and $k \oplus \Omega(k)$, respectively, by Proposition 1.8 of [5].

While the restrictions given in Theorems 1.2 and 1.3 do not give sharp criteria for the existence of kE -modules of stable constant Jordan type $[a][b]$, they do yield improvements on the bounds given by Theorem 1.1 on the values of p and r for which there do not exist modules of stable constant Jordan type $[a][b]$ in many cases. In the final section we also consider modules of stable constant Jordan type $[3][2][1]$ to illustrate how our technique can be used to study kE -modules with more than two non-projective blocks.

2. VECTOR BUNDLES ON PROJECTIVE SPACE

In this section we introduce concepts from the theory of vector bundles on projective space that will be used in proving Theorems 1.2 and 1.3. The exposition closely follows that of Benson and Pevtsova [3], where much of the material was introduced.

We first consider the projective space $\mathbb{P}^{r-1} = \text{Proj } k[Y_1, \dots, Y_r]$, where Y_i are the homogeneous coordinate functions. We denote by \mathcal{O} the structure sheaf on \mathbb{P}^{r-1} , and for $j \in \mathbb{Z}$ we let $\mathcal{O}(j)$ denote the j th Serre twist of \mathcal{O} . Each coordinate Y_i defines a sheaf morphism $\mathcal{O}(j) \rightarrow \mathcal{O}(j + 1)$ via multiplication of homogeneous rational functions.

For a finitely generated kE -module M , we consider the trivial sheaf

$$\tilde{M} = M \otimes_k \mathcal{O} = \mathcal{O}^{\oplus \dim_k M}$$

on \mathbb{P}^{r-1} . Note that the fibre of \tilde{M} at every point $\bar{\alpha} \in \mathbb{P}^{r-1}$ is simply M . Each $X_i \in kE$ defines a sheaf morphism $\tilde{M}(j) \rightarrow \tilde{M}(j+1)$ via the kE -action on M . Hence we obtain a family of morphisms

$$\begin{aligned} \theta_M : \tilde{M}(j) &\rightarrow \tilde{M}(j+1) \\ m \otimes f &\mapsto \sum_i X_i m \otimes Y_i f \end{aligned}$$

as defined in Friedlander and Pevtsova [6].

For $1 \leq i \leq p$, Benson and Pevtsova [3] define functors $\mathcal{F}_i : \text{mod}(kE) \rightarrow \text{Coh}(\mathbb{P}^{r-1})$ assigning to a finitely generated kE -module M the coherent sheaves

$$\mathcal{F}_i(M) = \frac{\text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}}{\text{Ker } \theta_M \cap \text{Im } \theta_M^i}.$$

It is also proved in [3] that M has constant Jordan type if and only if $\mathcal{F}_i(M)$ is a vector bundle for all $1 \leq i \leq p$. In particular, if M has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$, then $\mathcal{F}_i(M)$ is a vector bundle of rank a_i . We make repeated use of the following result.

Theorem 2.1 (Benson and Pevtsova [3]). *The trivial sheaf \tilde{M} has a filtration with filtered quotients $\mathcal{F}_i(M)(j)$ for $1 \leq j < i \leq p$.*

Recall that the Chow ring of \mathbb{P}^{r-1} is $A^*(\mathbb{P}^{r-1}) = \mathbb{Z}[h]/(h^r)$. We denote the Chern polynomial of a vector bundle \mathcal{F} on \mathbb{P}^{r-1} by

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F})h + \dots + c_{r-1}(\mathcal{F})h^{r-1} \in A^*(\mathbb{P}^{r-1}).$$

Here the $c_m(\mathcal{F}) \in \mathbb{Z}$ are the Chern numbers of \mathcal{F} . The Chern numbers of the j th twist of \mathcal{F} are calculated using the formula

$$c_m(\mathcal{F}(j)) = \sum_{l=0}^m j^l \binom{\text{rank}(\mathcal{F}) - m + l}{l} c_{m-l}(\mathcal{F})$$

as given in [7, Example 3.2.2]. The final result needed is the following.

Theorem 2.2 (Benson and Pevtsova [3]). *If \mathcal{F} is a vector bundle on \mathbb{P}^{r-1} , then*

$$c(\mathcal{F})c(\mathcal{F}(1)) \cdots c(\mathcal{F}(p-1)) \equiv 1 \pmod{(p, h^{p-1})}$$

as elements of $\mathbb{Z}[h]/(h^r)$.

3. THE MAIN RESULT, $a \neq b$

Proof of Theorem 1.2. Suppose M has stable constant Jordan type $[a][b]$ and that $a \neq p-b$. Since $\mathcal{F}_a(M)$ and $\mathcal{F}_b(M)$ are vector bundles of rank one, their Chern polynomials are of

the form $c(\mathcal{F}_a(M)) = 1 + \alpha h$ and $c(\mathcal{F}_b(M)) = 1 + \beta h$, respectively. The trivial sheaf \tilde{M} is a direct sum of copies of \mathcal{O} , hence $c(\tilde{M}) = 1$. It follows from Theorems 2.1 and 2.2 that

$$1 = c(\tilde{M}) \equiv \prod_{n=0}^{a-1} (1 + (\alpha + n)h) \prod_{m=0}^{b-1} (1 + (\beta + m)h) \pmod{(p, h^{p-1})}$$

since Chern polynomials are multiplicative over short exact sequences. Because $p \geq 5$ and $r \geq 4$, the coefficients on h , h^2 and h^3 must all be divisible by p . For convenience, we normalize by setting $\alpha' = \alpha + \frac{1}{2}(a-1)$ and $\beta' = \beta + \frac{1}{2}(b-1)$. The coefficient on h is $a\alpha' + b\beta'$ so that

$$(1) \quad \beta' \equiv -\frac{a}{b}\alpha' \pmod{p}.$$

The coefficient on h^2 is

$$\begin{aligned} & \sum_{-\frac{a-1}{2} \leq n < m \leq \frac{a-1}{2}} (\alpha' + n)(\alpha' + m) + \sum_{\substack{-\frac{a-1}{2} \leq n \leq \frac{a-1}{2} \\ -\frac{b-1}{2} \leq m \leq \frac{b-1}{2}}} (\alpha' + n)(\beta' + m) \\ & + \sum_{-\frac{b-1}{2} \leq n < m \leq \frac{b-1}{2}} (\beta' + n)(\beta' + m) = -\frac{a(a+b)}{2b}\alpha'^2 - \frac{(a+b)(a^2 + b^2 - ab - 1)}{24} \end{aligned}$$

after substituting (1). Since $a \neq p - b$, this implies

$$(2) \quad \alpha'^2 \equiv -\frac{b(a^2 + b^2 - ab - 1)}{12a} \pmod{p}.$$

The coefficient on h^3 is

$$\begin{aligned} & \sum_{-\frac{a-1}{2} \leq n < m < l \leq \frac{a-1}{2}} (\alpha' + n)(\alpha' + m)(\alpha' + l) + \sum_{-\frac{b-1}{2} \leq n < m < l \leq \frac{b-1}{2}} (\beta' + n)(\beta' + m)(\beta' + l) \\ & + \sum_{\substack{-\frac{a-1}{2} \leq n < m \leq \frac{a-1}{2} \\ -\frac{b-1}{2} \leq l \leq \frac{b-1}{2}}} (\alpha' + n)(\alpha' + m)(\beta' + l) + \sum_{\substack{-\frac{b-1}{2} \leq n < m \leq \frac{b-1}{2} \\ -\frac{a-1}{2} \leq l \leq \frac{a-1}{2}}} (\alpha' + l)(\beta' + n)(\beta' + m) \\ & = -\frac{a(a+b)(a-b)}{3b^2}\alpha'^3 + \frac{a(a+b)(a-b)}{12}\alpha' \end{aligned}$$

after substituting (1). Since $a \neq b$ and $a \neq p - b$, we have

$$\alpha'^3 - \frac{1}{4}b^2\alpha' \equiv 0 \pmod{p}$$

so that $\alpha' \equiv 0$ or $\pm \frac{1}{2}b \pmod{p}$. Substituting $\alpha' \equiv 0$ into (2) yields

$$a^2 + b^2 - ab - 1 \equiv 0 \pmod{p},$$

and substituting $\alpha' \equiv \pm \frac{1}{2}b$ into (2) yields

$$a^2 + b^2 + 2ab - 1 \equiv 0 \pmod{p}.$$

The last congruence has solutions $b \equiv -a \pm 1 \pmod{p}$, completing the proof. \square

Corollary 3.1. *Let $p \geq 5$ and $r \geq 4$. If a kE -module has stable constant Jordan type $[a][1]$ with $2 \leq a \leq p-1$, then $a = p-1$ or $a = p-2$.*

Proof. Conditions (i) and (ii) of Theorem 1.2 imply the result. If condition (iii) holds, then $a^2 - a \equiv 0 \pmod{p}$, forcing $a = 1$ or $a = p$, a contradiction. \square

Corollary 3.2. *If $p \geq 5$, $r \geq 4$ and $2 \leq a \leq p-3$, then there does not exist a kE -module of stable constant Jordan type $[a][1]$.*

Note that Theorem 1.1 guarantees the stable Jordan type $[a][1]$ is not realized provided $r \geq a+2$ and $p \geq a+3$, whereas Corollary 3.2 allows us to deduce the same statement whenever $r \geq 4$. Hence we obtain new information for the cases $4 \leq r \leq a+1$.

Example 3.3. *If $p \geq 7$ and $r \geq 4$, then there does not exist a kE -module of stable constant Jordan type $[3][1]$.*

The above restriction technique can be applied to many other stable Jordan types $[a][b]$. For example we have the following.

Proposition 3.4. *If $p \geq 7$ and $r \geq 4$, then there does not exist a kE -module of stable constant Jordan type $[3][2]$.*

Proof. Since $p \geq 7$, conditions (i) and (ii) in Theorem 1.2 are clearly not satisfied. Hence if such a module exists, then condition (iii) must hold so that $6 \equiv 0 \pmod{p}$. It follows that $p = 2$ or 3 , a contradiction. \square

We end this section by comparing Theorem 1.2 with the following conjecture of Suslin, which appeared in [5].

Conjecture 3.5 (Suslin [5]). *If a kE -module has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ and $a_j \neq 0$ for some $2 \leq j \leq p-1$, then $a_{j-1} \neq 0$ or $a_{j+1} \neq 0$.*

We now wish to calculate which stable Jordan types $[a][b]$ would be excluded from being realized by Conjecture 3.5 in light of the restrictions given by Theorem 1.2.

Proposition 3.6. *Suppose $p \geq 5$, $r \geq 4$ and that Conjecture 3.5 holds. If a kE -module has stable constant Jordan type $[a][b]$ with $a \neq b$, then this stable Jordan type is either $[p-1][1]$ or $\left[\frac{p+1}{2}\right] \left[\frac{p-1}{2}\right]$.*

Proof. Suppose there is a module of stable constant Jordan type $[a][b]$ and that this stable Jordan type is not $[p-1][1]$ or $\left[\frac{p+1}{2}\right] \left[\frac{p-1}{2}\right]$. By Conjecture 3.5, we must have $b = a \pm 1$. Without loss of generality, assume $2 \leq a \leq p-2$. Since $p \geq 5$, conditions (i) and (ii) of Theorem 1.2 cannot hold. So assume condition (iii) holds, that is,

$$a^2 + b^2 - ab - 1 \equiv 0 \pmod{p}.$$

Then

$$0 \equiv a^2 + (a \pm 1)^2 - a(a \pm 1) - 1 = a^2 \pm a$$

so that $a \equiv \pm 1 \pmod{p}$, a contradiction. \square

4. THE MAIN RESULT, $a = b$

Proof of Theorem 1.3. Suppose M has stable constant Jordan type $[a]^2$. Let $u = c_1(\mathcal{F}_a(M))$ and $v = c_2(\mathcal{F}_a(M))$. For $n \in \mathbb{Z}$, the formula for Chern polynomials of twists gives us

$$c_1(\mathcal{F}_a(M)(n)) = u + 2n \quad \text{and} \quad c_2(\mathcal{F}_a(M)(n)) = v + un + n^2.$$

Hence as in the proof of Theorem 1.2, we have

$$\prod_{n=0}^{a-1} (1 + (u + 2n)h + (v + un + n^2)h^2) \equiv 1 \pmod{(p, h^{p-1})}.$$

Since $p \geq 7$ and $r \geq 5$, the coefficients on h , h^2 , h^3 and h^4 must all be divisible by p . The coefficient on h is $au + a(a-1)$, so $u \equiv -(a-1) \pmod{p}$. The coefficient on h^2 is

$$\sum_{n=0}^{a-1} (v + un + n^2) + \sum_{0 \leq n < m \leq a-1} (u + 2n)(u + 2m) = av - \frac{1}{6}a(a-1)(2a-1)$$

after substituting $u \equiv -(a-1)$. Hence $v \equiv \frac{1}{6}a(a-1)(2a-1)$. The coefficient on h^3 is

$$\sum_{\substack{0 \leq n, m \leq a-1 \\ n \neq m}} (v + un + n^2)(u + 2m) + \sum_{0 \leq n < m < l \leq a-1} (u + 2n)(u + 2m)(u + 2l).$$

Unfortunately, after substituting $u \equiv -(a-1)$, this expression is identically zero in v . Hence we are forced to consider the coefficient on h^4 , which is calculated using

$$\begin{aligned} & \sum_{0 \leq n < m \leq a-1} (v + un + n^2)(v + um + m^2) + \sum_{\substack{0 \leq n, m, l \leq a-1 \\ m \neq n \neq l \\ m < l}} (v + un + n^2)(u + 2m)(u + 2l) \\ & + \sum_{0 \leq n < m < l < k \leq a-1} (u + 2n)(u + 2m)(u + 2l)(u + 2k). \end{aligned}$$

Substituting $u \equiv -(a-1)$ and $v \equiv \frac{1}{6}a(a-1)(2a-1)$, this yields

$$\frac{1}{360}a(a+1)(a-1)(2a+1)(2a-1),$$

which is congruent to 0 (mod p). Since $2 \leq a \leq p-2$, we must have $2a+1 \equiv 0$ or $2a-1 \equiv 0$ so that $a = \frac{p-1}{2}$ or $a = \frac{p+1}{2}$. \square

Again, Theorem 1.3 gives different information from that given by Theorem 1.1. In particular, the former rules out the realization of the stable Jordan type $[3]^2$ for $p \geq 11$ and $r \geq 5$, whereas the latter only does this for $r \geq 7$.

5. NON-REALIZATION OF THE STABLE JORDAN TYPE $[3][2][1]$ IN CERTAIN CASES

We conclude with an example of how the techniques used in proving Theorems 1.2 and 1.3 can be used to obtain restrictions on the occurrence of larger stable Jordan types.

Theorem 5.1. *If $p \geq 11$ and $r \geq 6$ or $p \geq 13$ and $r \geq 5$, then there does not exist a kE -module of stable constant Jordan type $[3][2][1]$.*

Proof. Using the same technique as before, we assume M is a module with stable constant Jordan type $[3][2][1]$ and write

$$(1 + \alpha h)(1 + \beta h)(1 + (\beta + 1)h)(1 + \gamma h)(1 + (\gamma + 1)h)(1 + (\gamma + 2)h) \equiv 1 \pmod{(p, h^{p-1})}$$

where $\alpha = c_1(\mathcal{F}_1(M))$, $\beta = c_1(\mathcal{F}_2(M))$ and $\gamma = c_1(\mathcal{F}_3(M))$. The coefficient on h is $\alpha + 2\beta + 3\gamma + 4$, which in any case must be congruent to 0 (mod p). Hence $\alpha \equiv -2\beta - 3\gamma - 4 \pmod{p}$. Substituting this and calculating the coefficient on h^3 yields a factorization

$$-(\beta + \gamma + 1)(\beta + \gamma + 2)(2\beta + 8\gamma + 9) \equiv 0 \pmod{p},$$

hence $\beta \equiv -\gamma - 1$, $-\gamma - 2$ or $-4\gamma - \frac{9}{2}$. Substituting $\beta \equiv -\gamma - 1$ into the expression for the coefficient on h^2 yields

$$-3\gamma^2 - 6\gamma - 5 \equiv 0 \pmod{p}$$

so that

$$\gamma \equiv -1 \pm \frac{1}{3}u \pmod{p},$$

where $u \in \mathbb{Z}$ is an integer such that $u^2 \equiv -6$. Substituting this and $\beta \equiv -\gamma - 1$ into the expression for the coefficient on h^4 gives us $\frac{7}{3}$, which must be congruent to 0 (mod p). But this contradicts our assumption that $p \geq 11$. Similarly, substituting $\beta \equiv -\gamma - 2$ into the expression for the coefficient on h^2 again yields

$$-3\gamma^2 - 6\gamma - 5 \equiv 0 \pmod{p},$$

and substituting these results and calculating the coefficient on h^4 again gives us $\frac{7}{3}$, a contradiction. Hence we are left with the case $\beta \equiv -4\gamma - \frac{9}{2}$. Substituting this and calculating the coefficient on h^2 yields

$$-30\gamma^2 - 60\gamma - \frac{125}{4} \equiv 0 \pmod{p}$$

so that

$$\gamma \equiv -1 \pm \frac{1}{12}u \pmod{p}.$$

Substituting this and $\beta \equiv -4\gamma - \frac{9}{2}$ into the expression for the coefficient on h^4 yields $-\frac{77}{192}$, which is never congruent to zero so long as $p \neq 7$ or 11. If we demand that $p \geq 11$ and $r \geq 6$ however, then the coefficient on h^5 must be congruent to 0 (mod p), and one readily calculates this to be $\pm \frac{35}{96}u$, which is impossible. This completes the proof. \square

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF ABERDEEN, ABERDEEN AB24 3UE