ASSIGNMENT 6, DUE WEDNESDAY 7 MARCH

1. Write down an inductive definition of a^b in the Peano axiom system.

2. Use the inductive definition of multiplication in the Peano system to prove that multiplication of natural numbers is associative.

You may now use standard properties of addition and multiplication in the remainder of the assignment.

3. Use the principle of mathematical induction to prove that

$$0 + 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all natural numbers n.

4. Use the principle of mathematical induction to prove that if a and b are natural numbers and $b \neq 0$, then there exist natural numbers q and r such that

$$a = bq + r \qquad \text{and} \qquad 0 \leq r < b.$$

(Hint: Induct on a.)

5. Determine what is wrong with the following.

Theorem. *Every car has the same color.*

Proof. We proceed by induction on the number of cars. If there is only one car, then clearly every car has the same color. For the induction step, let $n \geq 1$ and assume that if a collection of cars has size n, then all of its members share the same color. Now suppose that there are $n+1$ cars. Label them

$$c_1, c_2, \ldots, c_n, c_{n+1}.$$

By the induction hypothesis, all of the cars $c_1, \ldots, c_n$ have the same color, say color X. Similarly, all of the cars $c_2, \ldots, c_{n+1}$ have the same color, say color Y. Since c_2 has both color X and color Y, we must have $X = Y$, thus $c_1, \ldots, c_{n+1}$ all have color X. □

6. Use induction to prove for all $n \geq 5$ that $2^n > n^2$.