# HOW TO PROVE THE STATEMENT $(\forall n \in \mathbb{N})P(n)$ BY INDUCTION

This brief document outlines the minimal sorts of things that need to appear in proofs by induction. I will be looking for each of these things when grading your homework and exams.

In both forms of induction, it is important that you identify which open sentence $P(n)$ you are trying to prove for all natural numbers $n$. That said, the actual notation '$P(n)$' does not need to appear anywhere in your proof.

**Using the PMI.**

- Say, 'We proceed by induction on $n$.' This lets the reader know that they should expect a proof by induction.
- Prove that $P(0)$ is true using some appropriate calculation. (If the problem asks you to prove $P(n)$ for all $n$ greater than or equal to some natural number $n_0$ other than 0, for example, $n_0 = 1$, replace 0 by $n_0$ in this and the following step.)
- Say something like, 'Thus the statement holds for $n = 0$.'
- You have completed the base step. Skip a line and start a new paragraph.
- Say, 'Now let $n$ be a natural number and assume that [*state* $P(n)$ *here*].' This is the inductive hypothesis, the most important statement in the proof.
- Show that $P(n+1)$ is also true using some appropriate calculation.
- Say, 'By the PMI, this shows that [*state* $P(n)$ *here*] for every natural number $n$.'

**Example.** For every natural number $n \geq 6$, we have $n^3 < n!$.

*Proof.* We proceed by induction on $n$. We have

$$6^3 = 216 < 720 = 6!,$$

thus the statement holds for $n = 6$.

Now let $n$ be a natural number and assume that $n^3 < n!$. Then

$$\begin{aligned}(n+1)^3 &= n^3 + 3n^2 + 3n + 1 \\ &< n^3 + 3n^2 + 3n + n && \text{since } n > 1 \\ &= n^3 + 3n^2 + 4n \\ &< n^3 + 3n^2 + n^2 && \text{since } n > 4 \\ &= n^3 + 4n^2 \\ &< n^3 + n^3 && \text{since } n > 4 \\ &= 2n^3 \\ &< (n+1)n^3 && \text{since } n > 1 \\ &< (n+1)n! && \text{by the IH} \\ &= (n+1)!.\end{aligned}$$

By the PMI, this shows that $n^3 < n!$ for every natural number $n \geq 6$. □

**Remark.** In the above proof, the base case was for $n_0 = 6$.

**Using the PCI.**

- Say, 'We proceed by complete induction on $n$.' This lets the reader know that they should expect an induction proof involving possibly more than one base case.

- Prove that $P(0)$ is true using some appropriate calculation. (If the problem asks you to prove $P(n)$ for all $n$ greater than or equal to some natural number $n_0$ other than 0, for example, $n_0 = 1$, replace 0 by $n_0$ in this and the following step.

- Say something like, 'Thus the statement holds for $n = 0$.'

- **If necessary,** show that $P(1)$ is true using a valid calculation, then say, 'Thus the statement holds for $n = 1$.' For example, when proving theorems about Fibonacci sequences, there are usually two base cases needed. There might be several other base cases you have to address, depending on how much information the induction step requires.

- Continue in this way until you have established all necessary base cases. Then skip a line.

- Say, 'Now let $n \geq r$ [*here* $r$ *is the highest base case you needed to prove above*] be a natural number and assume for every natural number $0 \leq k \leq n$ that [*state* $P(k)$ *here*].'' (If the problem asks you to prove $P(n)$ for all $n$ greater than or equal to some natural number $n_0$ other than 0, replace 0 by $n_0$ here.)

- Show that $P(n+1)$ is also true using some appropriate calculation.

- Say, 'By the PCI, this shows that [*state* $P(n)$ *here*] for every natural number $n$.'

**Example.** Every natural number $n \geq 23$ can be written in the form $n = 3s + 4t$ where $s$ and $t$ are integers satisfying $s \geq 3$ and $t \geq 2$.

*Proof.* We proceed by complete induction on $n$. We have

$$23 = 3 \cdot 5 + 4 \cdot 2, \qquad 24 = 3 \cdot 4 + 4 \cdot 3 \qquad \text{and} \qquad 25 = 3 \cdot 5 + 4 \cdot 2,$$

thus the statement holds for $n = 23$, 24 and 25.

Now let $n \geq 25$ be a natural number and assume for every natural number $23 \leq k \leq n$ that $k$ may be written as $k = 3s + 4t$, where $s \geq 3$ and $t \geq 2$ are integers (that depend on $k$). Because $n \geq 25$, we have $n - 2 \geq 23$. By the inductive hypothesis, the statement holds for $k = n - 2$, i.e., there exist integers $s \geq 3$ and $t \geq 2$ such that $n - 2 = 3s + 4t$. Then

$$n + 1 = (n - 2) + 3 = 3s + 4t + 3 = 3(s + 1) + 4t.$$

By the PCI, this shows that the statement holds for every natural number $n \geq 23$. □

**Remark.** In the above proof, it was essential that we established three base cases. To see why, suppose that $n$ in the inductive step were 27. In the proof of the inductive step, we would have needed to know that the statement was true for $n - 2 = 27 - 2 = 25$.