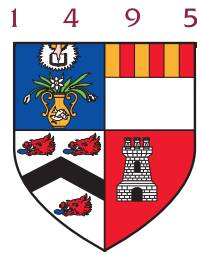


Some results on modules of constant Jordan type
for
elementary abelian p -groups



A thesis presented for the degree of
Doctor of Philosophy
at the University of Aberdeen

Shawn Baland
BA (U. Colorado, Boulder)

2012

Declaration

I declare that I have composed this thesis myself, that it has not been accepted in any previous application for a degree, and that the work contained herein is my own. I also declare that all quotations have been distinguished by quotation marks, and that all sources of information have been specifically acknowledged.

Acknowledgements

I would first like to thank my supervisor, Dave Benson, for his expertise and generosity with his time. It was an honour to be your student. I would also like to thank the members of the Institute of Mathematics for welcoming me into their department and allowing me to be part of their working family.

There were several people in the department with whom I had discussions that contributed to this work or my overall knowledge. In particular, I am grateful to Markus Linckelmann, Radha Kessar, Michał Stolorz and Aaron Chan. In addition, I would like to thank Michael Weiss and Filipp Levikov for electing to eat lunch at a reasonable hour. My warmest thanks also go to anybody who bought me coffee at the Hub on any occasion.

I was the recipient of some rather non-trivial funding whilst at Aberdeen. I am eternally grateful to Geoffrey Robinson for his profound generosity. My appreciation also goes to the Colorado Alpha Chapter of Phi Beta Kappa for partially funding my postgraduate research through the Crisp Fellowship.

Finally, I would like to thank Richard Green in Boulder for fostering my love of algebra, and for much encouragement throughout my undergraduate and postgraduate years.

Dedication

To Lyndsie and Stoney Pants, with all my love.

Summary

Let E be an elementary abelian p -group of rank r and let k be an algebraically closed field of characteristic p . We investigate finitely generated kE -modules of stable constant Jordan type $[a][b]$ with $1 \leq a, b \leq p - 1$ using the functors \mathcal{F}_i from finitely generated kE -modules to vector bundles on the projective space \mathbb{P}^{r-1} constructed by Benson and Pevtsova. In particular, we study relations on the Chern numbers of the trivial bundle \widetilde{M} to obtain restrictions on a and b for sufficiently large ranks and primes.

We then study kE -modules with the constant image property and define the constant image layers of a module with respect to its maximal submodule having the constant image property. We prove that almost all such subquotients are semisimple. Focusing on the class of W -modules in rank two, we also calculate the vector bundles $\mathcal{F}_i(M)$ for all W -modules M .

For E of rank two, we derive a duality formula for kE -modules M of constant Jordan type and their generic kernels $\mathfrak{K}(M)$. We use this to answer a question of Carlson, Friedlander and Suslin regarding whether or not the submodules $J^{-i}\mathfrak{K}(M)$ also have constant Jordan type for all $i \geq 0$. We show that this question has an affirmative answer whenever $p = 3$ or $J^2\mathfrak{K}(M) = 0$. We also show that it has a negative answer in general by constructing a kE -module M of constant Jordan type for $p \geq 5$ such that $J^{-1}\mathfrak{K}(M)$ does not have constant Jordan type.

Finally, we recall the proof of a theorem of Benson to show that if M is a kE -module of constant Jordan type containing no Jordan blocks of length one, then there always exist certain submodules of $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ having a particularly nice structure.

Contents

Chapter 1. Introduction	1
Chapter 2. Preliminaries	5
2.1. Nilpotent matrices and Jordan types	5
2.2. Elementary abelian p -group algebras	9
2.3. Representation type	11
2.4. Diagrams for modules	12
2.5. The generic Jordan type	15
2.6. Modules of constant Jordan type	19
2.7. Examples of modules of constant Jordan type	21
2.8. Some questions and conjectures	26
Chapter 3. Restrictions on Jordan types from vector bundles on projective space	28
3.1. Vector bundles on projective space	28
3.2. Trivial sheaves for kE -modules and the operator θ_M	30
3.3. The functors $\mathcal{F}_{i,j}$ and \mathcal{F}_i	33
3.4. Chern classes for vector bundles on \mathbb{P}^{r-1}	35
3.5. Benson's theorem on small Jordan types	36
3.6. Restrictions on the stable Jordan type $[a][b]$, $a \neq b$	37
3.7. Restrictions on the stable Jordan type $[a]^2$	40
3.8. Non-realisation of the stable Jordan type $[3][2][1]$ in certain cases	42
3.9. Discussion	43
Chapter 4. Constant image layers and vector bundles for W -modules	45
4.1. The constant image property	45
4.2. Constant image layers	47
4.3. W -modules and the constant image property in rank two	49
4.4. Vector bundles for W -modules	51
Chapter 5. Generic kernels and a question of Carlson, Friedlander and Suslin	53
5.1. Generic kernels	53
5.2. Generic kernels for modules of constant rank	55
5.3. The generic kernel filtration and the main question	56
5.4. Generic images	58
5.5. The constant kernel property	60
5.6. Duality for subquotients in the generic kernel filtration	62
5.7. Blocks of small length in certain subquotients	63
5.8. Question 5.10 in the cases $p = 3$ and $J^2\mathfrak{K}(M) = 0$	66
5.9. Some counterexamples to Question 5.10	68

Chapter 6. On the structure of the subquotient $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$	75
6.1. Benson's theorem on Rickard's conjecture	75
6.2. Diamond shaped submodules of $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$	78
Appendix A. Generic points in algebraic geometry	81
A.1. Generic points for affine varieties	81
A.2. Generic points in the language of schemes	83
Appendix B. Modules for \mathbb{Z}/p	85
Appendix C. A brief introduction to rank varieties	88
Bibliography	90

CHAPTER 1

Introduction

Elementary abelian p -groups have played an essential role in modular representation theory for over four decades now. This can be traced back to Quillen [33, 34], who showed for a finite group G and field k of characteristic p that the ring $H^*(G, k)$ modulo its nilpotent elements has Krull dimension equal to the maximal rank of any elementary abelian p -subgroup of G . A few years later, Dade [20, 21] classified the endotrivial modules for all finite abelian p -groups by using information coming from elementary abelian p -group algebras. Further motivation in this area came via Chouinard [18], who showed that if M is a module for a finite group G , then M is projective if and only if its restriction $M \downarrow_E$ to every elementary abelian subgroup E of G is projective.

Perhaps the greatest contribution in this area was that of Carlson [12, 13], who defined support varieties for modules over finite groups and, for an elementary abelian p -group E , constructed the rank variety $V_E^\sharp(M)$ of a kE -module M . Avrunin and Scott [2] later proved Carlson's conjecture that the rank variety and support variety coincide for kE -modules, effectively placing elementary abelian p -group representations at the centre of the rapidly emerging area of support theory.

The objects of interest in this thesis, namely modules of constant Jordan type, constitute a relatively new area in the modular representation of elementary abelian p -groups. That said, their definition arises naturally from the classical set-up.

Let k be an algebraically closed field of characteristic p and let $E \cong (\mathbb{Z}/p)^r$ be an elementary abelian p -group of rank r . Choosing generators g_1, \dots, g_r of E , we set $X_i = g_i - 1 \in kE$, and for each non-zero point $\alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$ we define the element $X_\alpha = \lambda_1 X_1 + \dots + \lambda_r X_r \in kE$. Because k has characteristic p , it follows that $X_\alpha^p = 0$, hence X_α acts on a finite dimensional kE -module M via a matrix whose Jordan canonical form contains Jordan blocks of length at most p . The *Jordan type* of X_α on M is the partition $[p]^{a_p}[p-1]^{a_{p-1}} \dots [1]^{a_1}$ of $\dim_k(M)$, where X_α acts on M via a_j Jordan blocks of length j . We say that M has *constant Jordan type* if the Jordan type of X_α on M is the same for all non-zero $\alpha \in \mathbb{A}^r(k)$. In this case, we call the list $[p-1]^{a_{p-1}} \dots [1]^{a_1}$ the *stable Jordan type* of M .

Modules of constant Jordan type were defined by Carlson, Friedlander and Pevtsova [16] in the context of finite group schemes and have been investigated more recently in [5, 6, 7, 9, 17] for elementary abelian p -groups. One of the primary objectives in studying these modules is to determine which stable Jordan types are realised as the stable Jordan types of kE -modules having constant Jordan type. Several conjectures have been made with the above goal in mind, although there has been little progress towards a full classification of realisable Jordan types.

An interesting development towards understanding modules of constant Jordan type was due to Benson and Pevtsova [9], who defined the functors \mathcal{F}_i for $1 \leq i \leq p$ from finitely generated kE -modules to coherent sheaves on the projective space \mathbb{P}^{r-1} . They also showed that a module M has constant Jordan type if and only if $\mathcal{F}_i(M)$ is an algebraic vector bundle on \mathbb{P}^{r-1} for all i . Benson [7] later used information coming from the Chern classes of the bundles $\mathcal{F}_i(M)$ to prove that if a kE -module has stable constant Jordan type $[p-1]^{a_{p-1}} \dots [1]^{a_1}$ and $\sum_{i=1}^{p-1} i \cdot a_i \leq \min(r-1, p-2)$, then $a_i = 0$ for all $2 \leq i \leq p-1$. Roughly speaking, this statement says that if a stable Jordan type contains some block of length greater than one and is ‘small’ relative to p and r , then it cannot be realised by a kE -module of constant Jordan type.

More recently, Carlson, Friedlander and Suslin [17] have investigated modules having the constant image property, i.e., those kE -modules M for which the image of X_α on M is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$. One of the key facts about the constant image property is that it implies the constant Jordan type property, although the converse does not hold.

In the case where E has rank two, the authors of [17] also defined the generic kernel $\mathfrak{K}(M)$ of a kE -module M , which can be characterised as the largest submodule of M having the constant image property. One of the striking features of the generic kernel is that, whenever M has constant rank (i.e., the rank of X_α on M is independent of the choice of non-zero $\alpha \in \mathbb{A}^2(k)$), the kernel of the action of X_α on M is contained in $\mathfrak{K}(M)$. Benson [6] used this property to prove that if a module (for any rank r) has constant Jordan type containing no blocks of length one, then the total number of Jordan blocks is divisible by p . This statement proves a special case of an unpublished conjecture of Rickard.

The results in this thesis are roughly divided into two parts. The first part expands on Benson and Pevtsova’s vector bundle approach to obtain new restrictions on small Jordan types, while the second investigates properties of the generic kernel for modules of constant rank over elementary abelian p -groups of rank two. The specific organisation is as follows.

In Chapter 2 we introduce the necessary background material for elementary abelian p -group algebras. We also provide a detailed exposition on generic Jordan types and then use this theory to give complete proofs of the main theorems in [16] concerning the behaviour of modules of constant Jordan type under common module theoretic operations. We then give some important examples of modules of constant Jordan type. Some of these appeared in [16], but we also provide a new family of examples in rank two whose constant Jordan types contain only blocks of length two and three. We close Chapter 2 with a discussion of some current problems in the area.

Chapter 3 begins with an overview of vector bundles on projective space. We then introduce the functors \mathcal{F}_i defined by Benson and Pevtsova and discuss their general properties. We also recall the theory of Chern classes for vector bundles on projective space and record a congruence formula for Chern polynomials of twists due to Benson and Pevtsova, which can be thought of as an analog of Fermat's little theorem. This will allow us to give a complete proof of Benson's theorem on small Jordan types. We then use the functors \mathcal{F}_i to obtain restrictions on the values of a and b for which there exists a kE -module of stable constant Jordan type $[a][b]$ for sufficiently large ranks and primes. In several cases, we show that these restrictions provide new bounds on p and r for which there does not exist a kE -module of stable constant Jordan type $[a][b]$. Finally, we use similar techniques to obtain restrictions on p and r for which there exists a kE -module of stable constant Jordan type $[3][2][1]$.

In Chapter 4 we transition to a discussion of modules with the constant image property. We show that every kE -module contains a unique submodule that is maximal with respect to the constant image property, and then define the *constant image layers* of a kE -module with respect to this submodule, analogously to how one defines the socle layers of modules over arbitrary rings. We prove for every kE -module that almost all of the constant image layers are semisimple. Restricting our attention to the case in which E has rank two, we then examine a special class of kE -modules with the constant image property called *W-modules*, which were also defined in [17]. After proving a general lemma about the behaviour of the functors \mathcal{F}_i with respect to modules with the constant image property, we then calculate the bundles $\mathcal{F}_i(M)$ for all W -modules M . As a corollary, we show that the functor \mathcal{F}_1 from modules of constant Jordan type to vector bundles on \mathbb{P}^1 is essentially surjective.

Chapter 5 seeks to answer Question 5.10, which was posed by Carlson, Friedlander and Suslin [17]. It asks the following: If a $k(\mathbb{Z}/p)^2$ -module M has constant Jordan type, is it true that $J^{-i}\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$?

Here $J = J(kE)$ denotes the Jacobson radical of kE , and $J^{-i}\mathfrak{K}(M)$ denotes the set of elements $m \in M$ for which $J^i m \subseteq \mathfrak{K}(M)$. We should point out that the original statement of Question 5.10 was given in terms of the submodules $X_1^{-i}\mathfrak{K}(M)$, but as we shall see in Proposition 5.8, these two formulations are equivalent. In addition to the above statement having been true in all prior known examples, motivation for Question 5.10 came from the observation that if M has constant Jordan type, then there exists a filtration

$$0 = J^p\mathfrak{K}(M) \subseteq J^{p-1}\mathfrak{K}(M) \subseteq \cdots \subseteq J^{-p+2}\mathfrak{K}(M) \subseteq J^{-p+1}\mathfrak{K}(M) = M$$

of M in which $J^i\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$ and for $i = -p + 1$.

Our results will show that Question 5.10 has an affirmative answer whenever $p = 3$ or $J^2\mathfrak{K}(M) = 0$. In proving this, we derive a particularly nice duality formula for the subquotients $J^a\mathfrak{K}(M)/J^b\mathfrak{K}(M)$, $a \leq b$, in the case where M has constant rank. We take special care to show that this result is independent of the choice of Hopf algebra structure on kE . Our approach will also lead us to investigate the number of blocks having small length in the Jordan type of X_α on $J^{-i}\mathfrak{K}(M)/J^j\mathfrak{K}(M)$, where α is a non-zero point in $\mathbb{A}^2(k)$, $i \geq 1$ and $j \geq 2$. Finally, we show that Question 5.10 has a negative answer in general by constructing a $k(\mathbb{Z}/p)^2$ -module such that $J^{-1}\mathfrak{K}(M)$ does not have constant Jordan type in the first interesting case, namely that in which $p \geq 5$, $J^2\mathfrak{K}(M) \neq 0$ and $J^3\mathfrak{K}(M) = 0$.

In Chapter 6 we recount the proof of the main theorem in Benson [6], which deals with modules of constant Jordan type containing no blocks of length one. In the rank two case, we use ideas from that proof to show for such kE -modules M that the subquotient $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ always contains submodules of constant Jordan type isomorphic to the examples introduced in Chapter 2.

In summary, the author's original results presented in this thesis are Theorems 2.22, 3.13 (this was due to Benson; we provide a new proof here), 3.14, 3.21, 3.23, Lemma 4.11, Theorem 4.12, Lemma 4.17, Theorems 4.18, 5.23, Lemma 5.24, Theorems 5.25, 5.26, 5.27, Lemma 5.28, Theorem 5.29, Example 5.30 and Theorem 6.2.

CHAPTER 2

Preliminaries

This chapter is dedicated to introducing background material from the modular representation theory of elementary abelian p -groups and modules of constant Jordan type. Our goal is to give a detailed summary of the main results of Carlson, Friedlander and Pevtsova [16]. For the representation theory of arbitrary finite groups, we refer the reader to Dornhoff [22, 23], Benson [3, 4] and Carlson [14]. A very nice treatment of modules over commutative rings can be found in Atiyah and MacDonal [1]. We close this chapter with a few interesting examples of modules of constant Jordan type and a discussion of some current problems in the area.

2.1. Nilpotent matrices and Jordan types

While it would be tempting to delve directly into the structures on which this thesis is based, we should note that modules of constant Jordan type have their roots in the theory of nilpotent matrices. Many of the notions we shall encounter when dealing with elementary abelian p -group representations are best introduced with respect to nilpotent matrices, and what's more, modules of constant Jordan type may only be defined intrinsically when one views them in this broader context. We begin with a discussion of orderings on nilpotent matrices.

If R is a ring, we denote by $\text{Mat}_n(R)$ the ring of $n \times n$ matrices over R . An element $A \in \text{Mat}_n(R)$ is *nilpotent* if there exists an integer $d > 0$ such that $A^d = 0$. The nilpotent elements of $\text{Mat}_n(R)$ may be partially ordered via the relation

$$A \succeq B \iff \text{rank}(A^s) \geq \text{rank}(B^s) \text{ for all } s \geq 0. \quad (2.1)$$

We obtain an equivalence relation from this ordering by setting $A \sim B$ if $A \succeq B$ and $B \succeq A$.

We shall focus mainly on the case where R is an algebra over an algebraically closed field k , e.g., the function field $k(t_1, \dots, t_r)$ in the algebraically independent variables t_1, \dots, t_r over k . For nilpotent matrices over any field, the partial ordering \succeq can be described in terms of Jordan canonical forms.

Let k be a field and let $A \in \text{Mat}_n(k)$. Recall that if k contains all eigenvalues of A , then A is conjugate in $\text{Mat}_n(k)$ to a block diagonal matrix

$$A' = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_t \end{pmatrix}$$

where each block J_i is of the form

$$J_i = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix},$$

λ being an eigenvalue of A . (Here we have omitted entries equal to zero.) The matrix A' is called the *Jordan canonical form* of A , and each block J_i is called a *Jordan block* of A' . Note that the Jordan canonical form of A is unique up to permutation of its Jordan blocks. Given a Jordan block J_i , we refer to the number of rows in J_i as its *length*.

Definition 2.1. If $A \in \text{Mat}_n(k)$ is nilpotent, then all of the eigenvalues of A are zero, hence the Jordan canonical form of A is completely determined by the lengths of its Jordan blocks. Listing the lengths of the Jordan blocks of A in descending order $d_1 \geq d_2 \geq \cdots \geq d_t$, we obtain a partition $[d_1][d_2] \cdots [d_t]$ of n . We call this partition the *Jordan type* of A .

Partitions of n are partially ordered via the dominance ordering. Specifically, if $\underline{d} = d_1 \geq d_2 \geq \cdots \geq d_t$ and $\underline{e} = e_1 \geq e_2 \geq \cdots \geq e_t$ are partitions of n , then \underline{d} is said to *dominate* \underline{e} , written $\underline{d} \geq \underline{e}$, if

$$\sum_{j=1}^i d_j \geq \sum_{j=1}^i e_j \quad \text{for all } 1 \leq i \leq t.$$

This allows us to partially order the nilpotent elements of $\text{Mat}_n(k)$ by defining $A \succeq B$ if the Jordan type of A dominates the Jordan type of B . It is well known that this ordering is equal to that defined by the condition (2.1). (For example, see Lemma 6.2.2 of [19].) In this case, we have $A \sim B$ if and only if A and B have the same Jordan canonical form, i.e., if and only if A and B are conjugate.

Our immediate goal is to use the above ordering to obtain some maximality criteria for Jordan types using the theory of generic points for affine varieties. (See Appendix A

for a discussion of generic points and generic properties.) The following is a variation of Lemma 1.2 of [26].

Proposition 2.2. *If k is an algebraically closed field, then the following are equivalent for matrices $A, A_1, \dots, A_r \in \text{Mat}_n(k)$.*

- (i) *If $k(t_1, \dots, t_r)$ is the function field in the algebraically independent variables t_1, \dots, t_r over k , then*

$$\text{rank}(A) = \text{rank}(A + t_1 A_1 + \dots + t_r A_r)$$

as matrices in $\text{Mat}_n(k(t_1, \dots, t_r))$.

- (ii) *For every field extension K/k and all $(\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(K)$,*

$$\text{rank}(A) \geq \text{rank}(A + \lambda_1 A_1 + \dots + \lambda_r A_r)$$

as matrices in $\text{Mat}_n(K)$.

- (iii) *For all $(\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$,*

$$\text{rank}(A) \geq \text{rank}(A + \lambda_1 A_1 + \dots + \lambda_r A_r).$$

- (iv) *The points $(\lambda_1, \dots, \lambda_r)$ for which*

$$\text{rank}(A) = \text{rank}(A + \lambda_1 A_1 + \dots + \lambda_r A_r)$$

form a dense open subset of $\mathbb{A}^r(k)$.

PROOF. We may think of $(t_1, \dots, t_r) \in \mathbb{A}^r(k(t_1, \dots, t_r))$ as the generic point of $\mathbb{A}^r(k)$ since, for any polynomial $f \in k[t_1, \dots, t_r]$, we have

$$f(\lambda_1, \dots, \lambda_r) = 0 \quad \text{for all } (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$$

if and only if $f = 0$. Setting $s = \text{rank}(A)$, the condition

$$\text{rank}(A + t_1 A_1 + \dots + t_r A_r) = s \tag{2.2}$$

is characterised in terms of the matrix minors of $A + t_1 A_1 + \dots + t_r A_r$, which are homogeneous polynomials in $k[t_1, \dots, t_r]$ by the formula for determinants. Specifically, (2.2) holds if and only if every $(s+1) \times (s+1)$ minor of $A + t_1 A_1 + \dots + t_r A_r$ is the zero polynomial, and some $s \times s$ minor of $A + t_1 A_1 + \dots + t_r A_r$ is non-zero. The proof now follows by applying Proposition A.2 to the set $T \subseteq k[t_1, \dots, t_r]$ of $s \times s$ minors of $A + t_1 A_1 + \dots + t_r A_r$. Note that the open subset in (iv) is non-empty since it contains the point $(0, \dots, 0) \in \mathbb{A}^r(k)$. \square

The proof of Proposition 2.2 can be easily applied to the matrix minors of powers of $A \in \text{Mat}_n(k)$. The following appeared as Proposition 1.3 of [26].

Proposition 2.3. *If k is an algebraically closed field, then the following are equivalent for commuting nilpotent matrices $A, A_1, \dots, A_r \in \text{Mat}_n(k)$.*

- (i) *If $k(t_1, \dots, t_r)$ is the function field in the algebraically independent variables t_1, \dots, t_r over k , then*

$$A \sim A + t_1 A_1 + \dots + t_r A_r$$

as matrices in $\text{Mat}_n(k(t_1, \dots, t_r))$.

- (ii) *For every field extension K/k and all $(\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(K)$,*

$$A \succeq A + \lambda_1 A_1 + \dots + \lambda_r A_r$$

as matrices in $\text{Mat}_n(K)$.

- (iii) *For all $(\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$,*

$$A \succeq A + \lambda_1 A_1 + \dots + \lambda_r A_r.$$

- (iv) *The points $(\lambda_1, \dots, \lambda_r)$ for which*

$$A \sim A + \lambda_1 A_1 + \dots + \lambda_r A_r$$

form a dense open subset of $\mathbb{A}^r(k)$.

Motivated by the previous two propositions, the following appeared as Definition 1.4 of [26].

Definition 2.4. Let $A, A_1, \dots, A_r \in \text{Mat}_n(k)$ be commuting nilpotent matrices over an algebraically closed field k . We say that A is *rank maximal* with respect to A_1, \dots, A_r if A, A_1, \dots, A_r satisfy the equivalent conditions of Proposition 2.2. Similarly, we say that A is *maximal* with respect to A_1, \dots, A_r if A, A_1, \dots, A_r satisfy the equivalent conditions of Proposition 2.3.

The following two theorems, which appear as Theorems 1.9 and 1.12 of [26], respectively, are the main results concerning rank maximality and maximality that will be needed in the sequel. The second will be especially important when proving the intrinsic nature of the constant Jordan type property in Section 2.6.

Theorem 2.5. *If k is an algebraically closed field and $A, A_1, \dots, A_r, B_1, \dots, B_r \in \text{Mat}_n(k)$ are commuting nilpotent matrices such that A is rank maximal with respect to A_1, \dots, A_r , then A is also rank maximal with respect to $A_1, \dots, A_r, A_1 B_1, \dots, A_r B_r$. Moreover, we have $\text{rank}(A) = \text{rank}(A + A_1 B_1 + \dots + A_r B_r)$.*

Theorem 2.6. *If k is an algebraically closed field and $A, A_1, \dots, A_r, B_1, \dots, B_r \in \text{Mat}_n(k)$ are commuting nilpotent matrices such that A is maximal with respect to*

A_1, \dots, A_r , then A is also maximal with respect to $A_1, \dots, A_r, A_1B_1, \dots, A_rB_r$. Moreover, we have $A \sim A + A_1B_1 + \dots + A_rB_r$.

2.2. Elementary abelian p -group algebras

Let p be a prime number and k an algebraically closed field of characteristic p . Throughout the rest of this thesis we consider an elementary abelian p -group E of rank r , that is,

$$E = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/p)^r,$$

where the g_i denote a fixed set of pairwise commuting generators of order p . To study finite dimensional kE -modules, it will be convenient to consider the elements $X_i = g_i - 1 \in kE$. Because k has characteristic p , we have $X_i^p = g_i^p - 1^p = 0$ for all i . It follows that kE is isomorphic to a truncated polynomial ring

$$kE \cong k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p),$$

where the k -algebra isomorphism maps each X_i to the coset containing t_i .

Note that any element $x \in kE$ can be written in the form $\lambda + f(X_1, \dots, X_r)$, where $\lambda \in k$ and $f \in k[t_1, \dots, t_r]$ is a polynomial with zero constant term. Since each X_i has p th power zero, we have $f(X_1, \dots, X_r)^p = 0$, hence $x^p = \lambda^p$. It follows that x is a zero divisor if and only if $\lambda = 0$. This shows that kE is a local ring with unique maximal ideal

$$J = J(kE) = (X_1, \dots, X_r).$$

In particular, every projective kE -module is free, thus the regular representation kE is the unique projective indecomposable kE -module up to isomorphism. By the usual correspondence between projective indecomposable and simple modules, this implies that the trivial module $k \cong kE/J$ is the unique simple kE -module up to isomorphism.

Whenever we consider a kE -module throughout this thesis, it will always be implied that the module in question is finitely generated, i.e., finite dimensional. As a matter of notation, if M is a kE -module and $x \in kE$, we denote by $\text{Ker}(x, M)$, $\text{Im}(x, M)$ and $\text{rank}(x, M)$ the kernel, image and rank of x when considered as a vector space endomorphism $M \rightarrow M$. Assuming we have chosen a basis of M , we denote by x_M the image of x under the matrix representation $kE \rightarrow \text{Mat}_{\dim_k(M)}(k)$ affording M .

By the above remarks, if M is a kE -module and $x \in J$, then x_M is a nilpotent matrix whose Jordan canonical form contains Jordan blocks of length at most p . We call the Jordan type of x_M the *Jordan type of x on M* (see Definition 2.1) and denote this by

$$\text{JType}(x, M) = [p]^{a_p}[p-1]^{a_{p-1}} \dots [1]^{a_1}$$

where, for $1 \leq j \leq p$, x_M contains a_j Jordan blocks of length j . It is clear from the definition that $\mathbf{JType}(x, M)$ is a partition of $\dim_k(M)$.

We now introduce a special family of elements in J that are parameterised by points in the affine space $\mathbb{A}^r(k)$. Specifically, for each $\alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$ we define

$$X_\alpha = \lambda_1 X_1 + \dots + \lambda_r X_r \in J.$$

Under the assignment $\alpha \mapsto X_\alpha + J^2$, we may identify $\mathbb{A}^r(k)$ with J/J^2 as k -vector spaces. This relationship will be of particular importance when we study projective $(r-1)$ -space in the context of kE -modules. We shall often find it necessary to restrict our attention to the elements X_α for which $\alpha \neq 0$. For such a point α , we define the element $g_\alpha = 1 + X_\alpha \in kE$. Because $X_\alpha^p = 0$ and $X_\alpha \neq 0$, we have $g_\alpha^p = 1$ and $g_\alpha^i \neq 1$ for all $i < p$. This shows that g_α has order p in the multiplicative group kE^\times . We call $\langle g_\alpha \rangle \cong \mathbb{Z}/p$ a *cyclic shifted subgroup* of kE .

A significant amount of the theory we investigate takes advantage of various k -Hopf algebra structures on kE . Recall that every counit for kE is necessarily a k -algebra homomorphism $kE \rightarrow k$. We have a vector space decomposition $kE = k \oplus J$, and all elements in J are nilpotent. This shows that there exists a unique counit for kE , namely the projection of kE onto the first summand. One readily verifies that this coincides with the augmentation map $\varepsilon: kE \rightarrow k$ given by $g \mapsto 1$ for all $g \in E$.

We draw particular attention to two common k -Hopf algebra structures on kE . The *group theoretic* Hopf algebra structure has comultiplication $\Delta: kE \rightarrow kE \otimes_k kE$ given by $g \mapsto g \otimes g$ for all $g \in E$, and antipode $\sigma: kE \rightarrow kE$ given by $g \mapsto g^{-1}$ for all $g \in E$. The *p -restricted Lie* Hopf algebra structure on kE has comultiplication $\tilde{\Delta}: kE \rightarrow kE \otimes_k kE$ given by $g - 1 \mapsto (g - 1) \otimes 1 + 1 \otimes (g - 1)$ for all $g \in E$, and antipode $\tilde{\sigma}: kE \rightarrow kE$ given by $g - 1 \mapsto -(g - 1)$ for all $g \in E$.

The above Hopf algebra structures yield two, generally different tensor product and dual kE -module structures. If M and N are finite dimensional kE -modules, we denote by $M \tilde{\otimes}_k N$ the k -vector space $M \otimes_k N$ equipped with the kE -module structure

$$(g - 1).(m \otimes n) = (g - 1)m \otimes n + m \otimes (g - 1)n \quad \text{for all } m \in M, n \in N, g \in E$$

induced by the Lie theoretic comultiplication $\tilde{\Delta}$, reserving the notation $M \otimes_k N$ for the kE -module structure

$$g.(m \otimes n) = gm \otimes gn \quad \text{for all } m \in M, n \in N, g \in E$$

induced by the group theoretic comultiplication Δ .

Similarly, if M is a finite dimensional kE -module, we denote by M^* the k -linear dual $\text{Hom}_k(M, k)$ equipped with the kE -module structure

$$((g-1).f)(m) = -f((g-1)m) \quad \text{for all } f \in \text{Hom}_k(M, k), g \in E, m \in M$$

induced by the Lie theoretic antipode $\tilde{\sigma}$, reserving the notation M^* for the dual kE -module structure

$$(g.f)(m) = f(g^{-1}m) \quad \text{for all } f \in \text{Hom}_k(M, k), g \in E, m \in M$$

induced by the group theoretic antipode σ .

The main distinction we wish to draw between the above module structures is that, in the Lie theoretic case, restriction to cyclic shifted subgroups commutes with taking tensor products, i.e., given kE -modules M and N we have

$$(M \tilde{\otimes}_k N) \downarrow_{\langle g_\alpha \rangle} \cong M \downarrow_{\langle g_\alpha \rangle} \tilde{\otimes}_k N \downarrow_{\langle g_\alpha \rangle} \quad \text{for all non-zero } \alpha \in \mathbb{A}^r(k).$$

This follows from the fact that kE , equipped with the p -restricted Lie Hopf algebra structure, is isomorphic to the universal enveloping algebra of a p -restricted Lie algebra with r generators, zero bracket and zero p th power map. In this case, the inclusion of a cyclic shifted subgroup into kE is induced by an inclusion of Lie algebras, hence is an inclusion of Hopf algebras.

2.3. Representation type

If k is an algebraically closed field, then by the trichotomy theorem of Drozd [24], any finite dimensional k -algebra A satisfies one of the following mutually exclusive conditions.

- (1) Finite representation type: There are only finitely many isomorphism classes of finite dimensional indecomposable left A -modules.
- (2) Tame representation type: There are infinitely many isomorphism classes of finite dimensional indecomposable left A -modules, but in any dimension they come in one parameter subfamilies, with finitely many exceptions.
- (3) Wild representation type: There exists a finite dimensional A - $k\langle X, Y \rangle$ -bimodule M that is free as a right $k\langle X, Y \rangle$ -module, such that the functor $M \otimes_{k\langle X, Y \rangle} -$ taking finite dimensional right $k\langle X, Y \rangle$ -modules to finite dimensional left A -modules preserves indecomposability and isomorphism classes. In this case, classifying the finite dimensional indecomposable A -modules would involve classifying pairs of square matrices under simultaneous conjugation, which is thought to be impossible.

For group algebras we have the following theorem of Bondarenko and Drozd [10].

Theorem 2.7. *Let G be a finite group and k an algebraically closed field of characteristic p .*

- (i) *If the Sylow p -subgroups of G are cyclic, then kG has finite representation type.*
- (ii) *If $p = 2$ and the Sylow p -subgroups of G are dihedral, semidihedral or generalised quaternion, then kG has tame representation type.*
- (iii) *In all other cases, kG has wild representation type.*

In terms of elementary abelian p -group algebras, this means that kE has finite representation type when $r = 1$, tame representation type when $p = r = 2$, and wild representation type in all other cases. (For the classification of the indecomposable $k(\mathbb{Z}/p)$ -modules, see Appendix B.) This means that in almost all cases we cannot hope to find a classification of the finite dimensional indecomposable kE -modules.

This obstruction to fully understanding the category $\mathbf{mod}(kE)$ often forces us to restrict our attention to certain subclasses of kE -modules that we do hope to classify, e.g., endotrivial modules. (See Dade [21].) Another common approach is to identify an invariant of a kE -module, and then try to study the behaviour of the invariant with respect to that of the module. A particularly fruitful example of this is Carlson's rank variety. (See Appendix C for a brief introduction.) As we shall see in the following sections, both of these ideas motivate the study of modules of constant Jordan type in the sense that we restrict our attention to a special subclass of kE -modules for which there is an associated invariant, namely the Jordan type of any element of $J \setminus J^2$ acting the module.

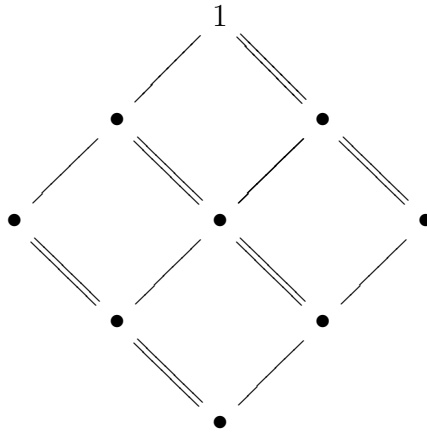
2.4. Diagrams for modules

The most useful tools for gaining intuition about representations of kE in low ranks are diagrams for modules. These were defined rigorously by Benson and Carlson [8], although the use of similar objects in the literature preceded their definition. For our purposes, a *module diagram* for a kE -module M consists of the following data.

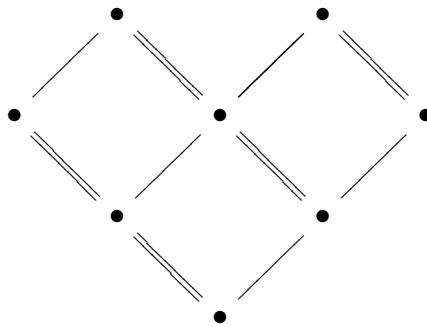
- (1) A finite directed graph with no loops whose vertex set is a fixed k -basis v_1, \dots, v_d of M .
- (2) There are r types of edges, r being the rank of E . An edge of type i has source u_j and target v_l if and only if $X_i v_j = v_l$ in M .
- (3) If there is no arrow of type i with source v_j , then $X_i v_j = 0$.

It follows from the definition that a kE -module is indecomposable if and only if its associated diagram is a connected graph. We note that most kE -modules do not have corresponding module diagrams, and even when such diagrams exist, they may not be easy to draw.

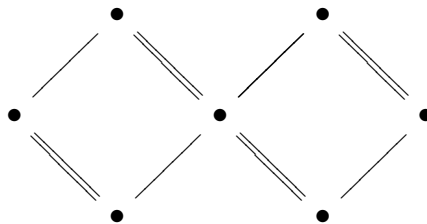
The diagrams we work with will almost exclusively be for modules in the case where E has rank two. An instructive example of such a diagram is that for the regular representation kE for $p = 3$.



Here X_1 acts via single edges downwards to the left and X_2 acts via double edges downwards to the right. For example, the middle vertex in the above diagram represents the element $X_1X_2 \in kE$. The submodule $\text{Rad}(kE)$ is represented by the diagram

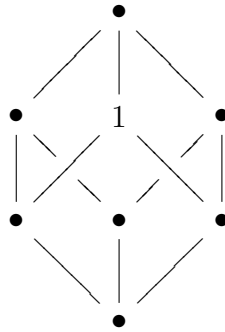


and $\text{Rad}(kE)/\text{Rad}^4(kE)$ is represented by the diagram



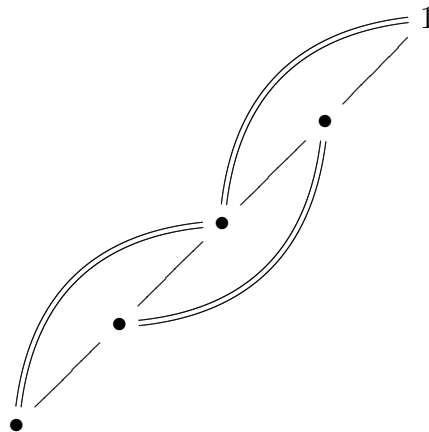
Note that the Jordan type of X_1 on the above modules can be read directly from the respective diagrams by counting the number of vertices in each path containing only X_1 -edges. For example, X_1 has Jordan type $[3]^3$ on kE , $[3]^2[2]$ on $\text{Rad}(kE)$ and $[3][2]^2$ on $\text{Rad}(kE)/\text{Rad}^4(kE)$. Similar statements hold for X_2 . In fact, we shall see in Section 2.7 that each of the above modules has constant Jordan type.

To see an example of a module diagram in rank three, we present the diagram for the regular representation kE for $p = 2$.



Here X_1 acts downwards to the left, X_2 acts downwards to the right and X_3 acts upwards. In particular, the socle of kE is spanned by the middle vertex in the third row. An attempt to draw a similar diagram for $p > 2$ quickly illustrates the limitations of diagrammatic techniques.

Returning to the rank two case, we now present a diagram representing the module $M = kE/(X_1^2 - X_2)$ for $p = 5$.



Here X_1 acts downwards via single edges and X_2 acts downwards via double edges. This module originally appeared as Example 2.3 of [26]. The authors of that paper observed that, although $X_1^2 - X_2 \equiv X_2 \pmod{J^2}$, $X_1^2 - X_2$ and X_2 have different Jordan types on M . In particular, $X_1^2 - X_2$ acts with Jordan type $[1]^5$, whereas X_2 acts with Jordan type $[3][2]$. (Note that this can be read directly from the diagram.)

One should compare this with the statement of Theorem 2.8, which shows that the above behaviour does not happen ‘generically’.

2.5. The generic Jordan type

In this section we introduce the concept of the *generic Jordan type* for a kE -module. This work was originally pioneered by Wheeler [35], and later strengthened by Friedlander, Pevtsova and Suslin [26], who showed that a module’s generic Jordan type is independent of the choice of generators of E . The material presented here will be essential in Section 2.6 when proving first properties of modules of constant Jordan type. The following appeared as Theorem 2.7 of [26].

Theorem 2.8. *Let M be a finite dimensional kE -module.*

- (i) *There exists a unique maximal Jordan type $J\text{Type}(x, M)$ with respect to the dominance ordering on partitions of $\dim_k(M)$, where x ranges over all elements of J .*
- (ii) *If $x \in J$ achieves the maximal Jordan type on M and $x - y \in J^2$, then y also achieves the maximal Jordan type on M .*
- (iii) *If M is not semisimple, then no element of J^2 achieves the maximal Jordan type on M .*

PROOF. If M is semisimple, then $JM = 0$ so that every element of J has the same Jordan type on M . So assume M is not semisimple. Let

$$(t_1, \dots, t_r) \in \mathbb{A}^r(k(t_1, \dots, t_r))$$

be the generic point of $\mathbb{A}^r(k)$ and define

$$X_{\text{gen}} = t_1 X_1 + \dots + t_r X_r \in kE \otimes_k k(t_1, \dots, t_r).$$

Writing $\bar{M} = M \otimes_k k(t_1, \dots, t_r)$, the matrix minors of $(X_{\text{gen}})_{\bar{M}}$ are polynomials in $k[t_1, \dots, t_r]$, hence by Proposition A.2 there exists a dense open subset $U \subseteq \mathbb{A}^r(k)$ such that $\alpha \in U$ if and only if $(X_\alpha)_{\bar{M}} \sim (X_{\text{gen}})_{\bar{M}}$. Fixing $\alpha \in U$, we have

$$(X_\alpha)_{\bar{M}} \sim (X_{\text{gen}})_{\bar{M}} \succeq (X_{\alpha+\beta})_{\bar{M}} = (X_\alpha)_{\bar{M}} + (X_\beta)_{\bar{M}}$$

for all $\beta \in \mathbb{A}^r(k)$. It follows that $(X_\alpha)_M \succeq (X_\alpha)_M + (X_\beta)_M$ for all $\beta \in \mathbb{A}^r(k)$, thus $(X_\alpha)_M$ is maximal with respect to the family of matrices $(X_1)_M, \dots, (X_r)_M$. Next observe that any $x \in J$ is of the form $x = X_\beta + X_1 y_1 + \dots + X_r y_r$ with $\beta \in \mathbb{A}^r(k)$ and $y_1, \dots, y_r \in J$, hence each y_i acts nilpotently on M . Applying Theorem 2.6 to

the family of nilpotent matrices $(X_1)_M, \dots, (X_r)_M, (y_1)_M, \dots, (y_r)_M$, we see that

$$(X_\alpha)_M \succeq (X_\beta + X_1 y_1 + \dots + X_r y_r)_M = x_M,$$

thus $\text{JType}(X_\alpha, M)$ dominates $\text{JType}(x, M)$ for all $x \in J$. This establishes (i).

To prove (ii), note that if $x - y \in J^2$, then as above we have $y = x + X_1 y_1 + \dots + X_r y_r$ where $y_1, \dots, y_r \in J$ act nilpotently on M . Because x achieves the maximal Jordan type on M , we also have $x_M \succeq x_M + (X_\beta)_M$ for all $\beta \in \mathbb{A}^r(k)$, hence x_M is maximal with respect to $(X_1)_M, \dots, (X_r)_M$. It now follows from the second part of Theorem 2.6 that

$$x_M \sim (x + X_1 y_1 + \dots + X_r y_r)_M = y_M$$

so that $\text{JType}(x, M) = \text{JType}(y, M)$.

(iii) now follows from (ii) by noting that if $x \in J^2$ achieves the maximal Jordan type on M , then this Jordan type is equal to that of $0 \in kE$, hence M must be semisimple. \square

Definition 2.9 (Friedlander, Pevtsova and Suslin [26]). If M is a kE -module, then in light of Theorem 2.8 (i) we define the *generic Jordan type* of M to be the maximal Jordan type achieved by any element of J acting on M . (This terminology is motivated by the fact that the maximal Jordan type on M is equal to that of the generic element X_{gen} on \bar{M} .) By the proof of the same theorem, the points $\alpha \in \mathbb{A}^r(k)$ for which X_α has the generic Jordan type on M form a dense open subset of $\mathbb{A}^r(k)$. We denote this subset by $U_{\text{gen}}(M)$.

The next proposition shows that the generic Jordan type is well behaved with respect to field extensions.

Proposition 2.10. *Let M be a kE -module and let K/k be any field extension. If $\alpha \in U_{\text{gen}}(M)$, then $(X_\alpha)_{M \otimes_k K} \succeq (X_\beta)_{M \otimes_k K}$ for all $\beta \in \mathbb{A}^r(K)$.*

PROOF. We have $(X_\alpha)_M \succeq (X_{\alpha+\beta})_M = (X_\alpha)_M + (X_\beta)_M$ for all $\beta \in \mathbb{A}^r(k)$. By condition (ii) of Proposition 2.3, this implies that

$$(X_\alpha)_{M \otimes_k K} \succeq (X_\alpha)_{M \otimes_k K} + (X_\beta)_{M \otimes_k K} \quad \text{for all } \beta \in \mathbb{A}^r(K).$$

Thus if $\gamma \in \mathbb{A}^r(K)$, then $(X_\alpha)_{M \otimes_k K} \succeq (X_\alpha)_{M \otimes_k K} + (X_{\gamma-\alpha})_{M \otimes_k K} = (X_\gamma)_{M \otimes_k K}$. \square

The following proposition relating the generic Jordan type to dualisation appeared as Proposition 5.2 of [16].

Proposition 2.11. *If M is a kE -module, then the generic Jordan type of M^* is equal to the generic Jordan type of M , and we have $U_{\text{gen}}(M) = U_{\text{gen}}(M^*)$.*

PROOF. If $\sigma: kE \rightarrow kE$ is the group theoretic antipode of kE (see Section 2.2), then for each $1 \leq i \leq r$ we have

$$X_i + \sigma(X_i) = (g_i - 1) + (g_i^{-1} - 1) = -(g_i - 1)(g_i^{-1} - 1) \in J^2,$$

hence $X_\alpha + \sigma(X_\alpha) \in J^2$ for all $\alpha \in \mathbb{A}^r(k)$. If X_α has the generic Jordan type on M , then it follows from the second part of Theorem 2.6 that $(X_\alpha)_M \sim \sigma(X_\alpha)_M$. We then have $(X_\alpha)_M \sim (X_\alpha)_{M^*}$ after noting that the representation affording M^* is obtained from the representation affording M by precomposing with σ and then composing with the transpose map. It now follows that X_α has the generic Jordan type on M^* , for otherwise we would have $(X_\beta)_{M^*} \succ (X_\alpha)_{M^*}$ for any $\beta \in U_{\text{gen}}(M^*)$, yielding

$$(X_\beta)_M \sim (X_\beta)_{M^*} \succ (X_\alpha)_{M^*} \sim (X_\alpha)_M,$$

a contradiction. \square

We shall also be interested in the behaviour of generic Jordan types with respect to direct sums. The following appeared as part of Theorem 4.7 of [26].

Proposition 2.12. *If M and N are kE -modules, then the generic Jordan type of $M \oplus N$ is equal to the union of the generic Jordan types of M and N as partitions. We also have $U_{\text{gen}}(M \oplus N) = U_{\text{gen}}(M) \cap U_{\text{gen}}(N)$.*

PROOF. Since $\mathbb{A}^r(k)$ is irreducible, $U_{\text{gen}}(M) \cap U_{\text{gen}}(N) \cap U_{\text{gen}}(M \oplus N)$ is non-empty, being an intersection of dense open subsets of $\mathbb{A}^r(k)$. Choosing a point α in this intersection, the equation $(X_\alpha)_{M \oplus N} = (X_\alpha)_M \oplus (X_\alpha)_N$ proves the first statement of the theorem, and the containment \supseteq in the second statement.

To prove the other containment, let $\alpha \in U_{\text{gen}}(M \oplus N)$ and suppose without loss of generality that X_α does not achieve the generic Jordan type on M . Choosing $\beta \in U_{\text{gen}}(M) \cap U_{\text{gen}}(N)$, we then have $(X_\beta)_M \succ (X_\alpha)_M$, hence there exists an integer $s > 0$ such that $\text{rank}(X_\beta^s, M) > \text{rank}(X_\alpha^s, M)$. We also have $(X_\beta)_N \succeq (X_\alpha)_N$ so that $\text{rank}(X_\beta^s, N) \geq \text{rank}(X_\alpha^s, N)$. It follows that

$$\begin{aligned} \text{rank}(X_\alpha^s, M \oplus N) &= \text{rank}(X_\alpha^s, M) + \text{rank}(X_\alpha^s, N) < \text{rank}(X_\beta^s, M) + \text{rank}(X_\beta^s, N) \\ &= \text{rank}(X_\beta^s, M \oplus N), \end{aligned}$$

contradicting the fact that $(X_\alpha)_{M \oplus N} \sim (X_\beta)_{M \oplus N}$. \square

The final result of this section relates the generic Jordan types of two kE -modules to that of their tensor product. It appeared as Proposition 4.4 of [26].

Proposition 2.13. *If M and N are kE -modules and $\alpha \in U_{\text{gen}}(M) \cap U_{\text{gen}}(N)$, then*

$$(M \otimes_k N) \downarrow_{\langle g_\alpha \rangle} \cong M \downarrow_{\langle g_\alpha \rangle} \otimes_k N \downarrow_{\langle g_\alpha \rangle}$$

as $k\langle g_\alpha \rangle$ -modules, and $\alpha \in U_{\text{gen}}(M \otimes_k N)$. In other words, the generic Jordan type of a tensor product is the tensor product of the respective generic Jordan types.

PROOF. We first recall the Lie theoretic tensor product $M \tilde{\otimes}_k N$ introduced in Section 2.2. Choosing a point $\alpha \in U_{\text{gen}}(M) \cap U_{\text{gen}}(N) \cap U_{\text{gen}}(M \otimes_k N)$, we have

$$(M \tilde{\otimes}_k N) \downarrow_{\langle g_\alpha \rangle} \cong M \downarrow_{\langle g_\alpha \rangle} \tilde{\otimes}_k N \downarrow_{\langle g_\alpha \rangle} \cong M \downarrow_{\langle g_\alpha \rangle} \otimes_k N \downarrow_{\langle g_\alpha \rangle} \quad (2.3)$$

by Corollary B.4.

Next recall from the proof of Theorem 2.8 that since X_α has the generic Jordan type on M , $(X_\alpha)_M$ is maximal with respect to the family of matrices $(X_1)_M, \dots, (X_r)_M$. By Proposition 2.3, this means that

$$(X_\alpha)_M \sim (X_\alpha)_M + t_1(X_1)_M + \dots + t_r(X_r)_M \quad (2.4)$$

for algebraically independent variables t_1, \dots, t_r over k . Similarly, we have

$$(X_\alpha)_N \sim (X_\alpha)_N + s_1(X_1)_N + \dots + s_r(X_r)_N \quad (2.5)$$

for another set of algebraically independent variables s_1, \dots, s_r over k . It follows that

$$\begin{aligned} (X_\alpha)_{M \tilde{\otimes}_k N} &= (X_\alpha)_M \otimes I_N + I_M \otimes (X_\alpha)_N \\ &\sim (X_\alpha)_M \otimes I_N + I_M \otimes (X_\alpha)_N \\ &\quad + t_1(X_1)_M \otimes I_N + \dots + t_r(X_r)_M \otimes I_N \\ &\quad + I_M \otimes s_1(X_1)_N + \dots + I_M \otimes s_r(X_r)_N. \end{aligned} \quad (2.6)$$

Indeed, if conjugation by A makes the matrices in (2.4) similar and conjugation by B makes the matrices in (2.5) similar, then conjugation by $A \otimes B$ makes the matrices in (2.6) similar. This shows that $(X_\alpha)_{M \tilde{\otimes}_k N}$ is maximal with respect to the family of matrices

$$(X_1)_M \otimes I_N, \dots, (X_r)_M \otimes I_N, I_M \otimes (X_1)_N, \dots, I_M \otimes (X_r)_N.$$

Now notice for each $1 \leq i \leq r$ that

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i + X_i \otimes X_i = \tilde{\Delta}(X_i) + X_i \otimes X_i.$$

Writing $\alpha = (\lambda_1, \dots, \lambda_r)$ with each $\lambda_i \in k$, it follows from the above equation and Theorem 2.6 that

$$(X_\alpha)_{M \otimes_k N} = (X_\alpha)_{M \otimes_k N} + \sum_{i=1}^r \lambda_i ((X_i)_M \otimes I_N)(I_M \otimes (X_i)_N) \sim (X_\alpha)_{M \otimes_k N}.$$

Combining this with (2.3) completes the proof. \square

2.6. Modules of constant Jordan type

By the discussion in Appendix B, if $\langle g \rangle \cong \mathbb{Z}/p$ is a cyclic group of order p and M is a finite dimensional $k\langle g \rangle$ -module, then the isomorphism type of M is completely determined by the Jordan type of $X = g - 1$ on M .

It follows that if M is a finite dimensional kE -module and α is a non-zero point in $\mathbb{A}^r(k)$, then since $X_\alpha = g_\alpha - 1$, the partition $\mathbf{JType}(X_\alpha, M)$ completely determines the isomorphism type of $M \downarrow_{\langle g_\alpha \rangle}$ as a $k\langle g_\alpha \rangle$ -module. In the general setting, this isomorphism type depends greatly on the choice of α . Viewed in this way, modules of constant Jordan type are precisely those kE -modules M for which $M \downarrow_{\langle g_\alpha \rangle}$ *does not* depend on α . They were first defined by Carlson, Friedlander and Pevtsova [16].

Definition 2.14. A finite dimensional kE -module M has *constant Jordan type* if $\mathbf{JType}(X_\alpha, M)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$, i.e., if and only if $\text{rank}(X_\alpha^s, M)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$ for all $s \geq 0$.

If M has constant Jordan type and $\mathbf{JType}(X_\alpha, M) = [p]^{a_p} \dots [1]^{a_1}$ for all non-zero $\alpha \in \mathbb{A}^r(k)$, we call the partition $[p]^{a_p} \dots [1]^{a_1}$ the *constant Jordan type* of M . In this case, the list $[p-1]^{a_{p-1}} \dots [1]^{a_1}$ is called the *stable constant Jordan type* of M . It is obtained by calculating the Jordan type of X_α on the non-projective part of $M \downarrow_{\langle g_\alpha \rangle}$ for any non-zero $\alpha \in \mathbb{A}^r(k)$.

Remark 2.15. The proof of Theorem 2.8 (i) shows that if M is not semisimple, then M has constant Jordan type if and only if $U_{\text{gen}}(M) = \mathbb{A}^r(k) \setminus \{0\}$.

It is not clear from Definition 2.14 that the constant Jordan type property is independent of the choice of generators g_1, \dots, g_r of E . The following proposition shows that this is indeed the case.

Proposition 2.16. *A finite dimensional kE -module M has constant Jordan type if and only if $\mathbf{JType}(x, M)$ is independent of $x \in J \setminus J^2$.*

PROOF. If $x \in J \setminus J^2$, then $x \equiv X_\alpha \pmod{J^2}$ for some non-zero $\alpha \in \mathbb{A}^2(k)$. If M has constant Jordan type, then by the proof of Theorem 2.8 (i), X_α must have

the generic Jordan type on M . It then follows by part (ii) of the same theorem that $x_M \sim (X_\alpha)_M$. This establishes the ‘only if’ direction, and the reverse implication is obvious. \square

Definition 2.17. A short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ of kE -modules is *locally split* if $0 \rightarrow M_1 \downarrow_{\langle g_\alpha \rangle} \rightarrow M_2 \downarrow_{\langle g_\alpha \rangle} \rightarrow M_3 \downarrow_{\langle g_\alpha \rangle} \rightarrow 0$ splits as a short exact sequence of $k\langle g_\alpha \rangle$ -modules for all non-zero $\alpha \in \mathbb{A}^r(k)$.

Carlson and Friedlander [15] have shown that the class of kE -modules of constant Jordan type forms a Quillen exact category $\mathbf{cJt}(kE)$, where the short exact sequences in $\mathbf{cJt}(kE)$ are defined to be the locally split short exact sequences in $\mathbf{mod}(kE)$. For a thorough discussion of exact categories, see Bühler [11].

The following theorem is due to Carlson, Friedlander and Pevtsova [16]. It shows that the constant Jordan type property remains invariant under many common operations on modules for Hopf algebras. Recall that if M is a kE -module, then the Heller shift $\Omega(M)$ is defined to be the kernel of the projective cover $P_M \rightarrow M$.

Theorem 2.18.

- (i) *If M is a kE -module of constant Jordan type, then $\Omega(M)$ and M^* have constant Jordan type.*
- (ii) *If M and N are kE -modules, then $M \oplus N$ has constant Jordan type if and only if M and N have constant Jordan type.*
- (iii) *If M and N are kE -modules of constant Jordan type, then $M \otimes_k N$ has constant Jordan type.*

PROOF. The second part (i) follows from Remark 2.15 and Proposition 2.11. Similarly, part (ii) follows from Remark 2.15 and Proposition 2.12. Part (iii) follows from Remark 2.15 and Proposition 2.13. We are therefore left to prove the first part of (i). Let α be a non-zero point in $\mathbb{A}^r(k)$ and consider the restriction

$$0 \longrightarrow \Omega(M) \downarrow_{\langle g_\alpha \rangle} \longrightarrow P_M \downarrow_{\langle g_\alpha \rangle} \longrightarrow M \downarrow_{\langle g_\alpha \rangle} \longrightarrow 0$$

of the short exact sequence defining $\Omega(M)$. Letting

$$0 \longrightarrow \Omega(M \downarrow_{\langle g_\alpha \rangle}) \longrightarrow P_{M \downarrow_{\langle g_\alpha \rangle}} \longrightarrow M \downarrow_{\langle g_\alpha \rangle} \longrightarrow 0$$

be the short exact sequence defining $\Omega(M \downarrow_{\langle g_\alpha \rangle})$, we have

$$\Omega(M) \downarrow_{\langle g_\alpha \rangle} \oplus P_{M \downarrow_{\langle g_\alpha \rangle}} \cong \Omega(M \downarrow_{\langle g_\alpha \rangle}) \oplus P_{M \downarrow_{\langle g_\alpha \rangle}}$$

by Schanuel's lemma. Shaving off the projective summand $P_{M \downarrow \langle g_\alpha \rangle}$ using the Krull-Schmidt theorem, we see that $\Omega(M) \downarrow_{\langle g_\alpha \rangle}$ is independent of α since $M \downarrow_{\langle g_\alpha \rangle}$ is independent of α . \square

It is easy to deduce from Propositions 2.11, 2.12 and 2.13 which constant Jordan types we obtain from the above operations. Observe that if M has constant Jordan type $[p]^{a_p}[p-1]^{a_{p-1}} \dots [i]^{a_i} \dots [1]^{a_1}$, then $\Omega(M)$ has constant Jordan type

$$[p]^{b_p}[p-1]^{a_1} \dots [i]^{a_{p-i}} \dots [1]^{a_{p-1}}$$

for some value of b_p . This follows from the way one computes Heller shifts of cyclic modules for $k(\mathbb{Z}/p)$.

2.7. Examples of modules of constant Jordan type

We now present some fairly naturally occurring examples of modules of constant Jordan type. Further examples will appear throughout Chapter 5.

First note by Dade's lemma (see Theorem C.1) that a kE -module is projective if and only if its restriction to every cyclic shifted subgroup is a projective $k(\mathbb{Z}/p)$ -module. It follows that the projective kE -modules are precisely those having constant Jordan type $[p]^n$ for some $n \geq 0$, and in this case we have $p^{r-1} \mid n$.

Closely related to the projective kE -modules, our next family of examples appeared in Proposition 2.1 of [16].

Proposition 2.19. *The subquotients J^a/J^b of kE have constant Jordan type for all $a \leq b$.*

PROOF. It suffices to show for every non-zero $\alpha \in \mathbb{A}^r(k)$ that

$$\text{JType}(X_\alpha, J^a/J^b) = \text{JType}(X_1, J^a/J^b).$$

Let $\phi: kE \rightarrow kE$ be an automorphism of kE induced from a matrix in $GL(r, k)$ sending the r -tuple $(1, 0, \dots, 0)$ to α , i.e., such that $\phi(X_1) = X_\alpha$. Under this automorphism we have $(J^a/J^b) \downarrow_{\langle g_\alpha \rangle} = \phi(J^a/J^b) \downarrow_{\langle \phi(1+X_1) \rangle} \cong (J^a/J^b) \downarrow_{\langle 1+X_1 \rangle}$. \square

Another natural family of examples were investigated in Section 5 of [16]. Recall that if G is a finite group, then a kG -module is *endotrivial* if $\text{End}_k(M) \cong k \oplus (\text{projective})$. Dade [21] classified the endotrivial modules for finite abelian p -groups. The following appeared as Theorem 10.1 of [21].

Theorem 2.20. *A module for a finite abelian p -group is endotrivial if and only if it is stably isomorphic to $\Omega^n(k)$ for some $n \in \mathbb{Z}$.*

Carlson, Friedlander and Pevtsova have provided an alternative characterisation of endotrivial modules for elementary abelian p -groups in terms of their stable Jordan types. The following appeared as Theorem 5.6 of [16].

Theorem 2.21. *A kE module is endotrivial if and only if it has stable constant Jordan type $[1]$ or $[p - 1]$.*

PROOF. If M is endotrivial, then $M \cong \Omega^n(k)$ for some n by Theorem 2.20. Theorem 2.18 (i) and its proof show that M has stable constant Jordan type $[1]$ for $p = 2$ or n even, and $[p - 1]$ for n odd. (This can also be shown without using Dade's classification, as was done in [16].)

Conversely, if M has stable constant Jordan type $[1]$, then M^* has stable constant Jordan type $[1]$ so that $\text{End}_k(M) \cong M^* \otimes_k M$ has stable constant Jordan type $[1]$. Consider the short exact sequence

$$0 \longrightarrow N \longrightarrow \text{End}_k(M) \xrightarrow{\text{Tr}} k \longrightarrow 0$$

where the map on the right is the matrix trace map and N is its kernel. Since $\dim_k(M)$ is not divisible by p , the map $k \rightarrow \text{End}_k(M)$ given by inclusion of scalar matrices is a splitting of the above sequence, hence $\text{End}_k(M) \cong k \oplus N$. Since $\text{End}_k(M)$ has stable constant Jordan type $[1]$, it follows from Theorem 2.18 (ii) that N has constant Jordan type containing only blocks of length p . By Dade's lemma, this shows that N is projective, thus M is endotrivial.

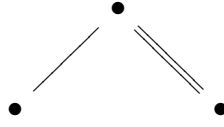
If M has stable constant Jordan type $[p - 1]$, then the restriction of $M^* \otimes_k M$ to a cyclic shifted subgroup is stably isomorphic to $J_{p-1} \otimes_k J_{p-1}$ as a $k(\mathbb{Z}/p)$ -module. (See Appendix B for an explanation of this notation.) We have $J_{p-1} \cong \Omega(k)$, and $\Omega(k) \otimes_k \Omega(k)$ is stably isomorphic to $\Omega^2(k) \cong k$. It follows that $\text{End}_k(M)$ has stable constant Jordan type $[1]$. The proof that M is endotrivial now follows exactly as above. \square

We end this section with a family of modules in rank two having constant Jordan type $[3]^a[2]^b$ for certain $a, b > 0$. These modules can only be described efficiently using module diagrams, and what's more, the verification of their constant Jordan types serves as an instructive exercise in the use of diagrammatic techniques.

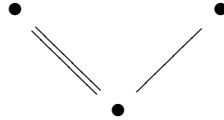
We begin by fixing an alphabet

$$\mathcal{A} = \{A, V, D_n \mid n \geq 2\}.$$

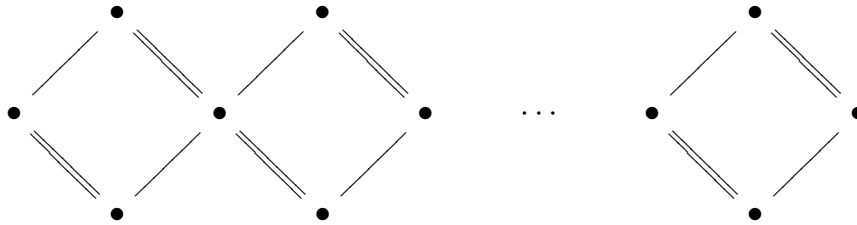
For each abstract word W over \mathcal{A} , we construct a kE -module $M(W)$ of radical length at most three in the following way. We associate to the letter A the graph



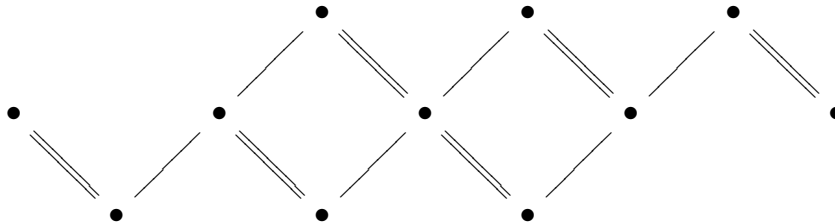
we associate to the letter V the graph



and for $n \geq 2$ we associate to the letter D_n the graph



where the number of middle vertices is equal to n . We then associate to the word W the graph obtained by connecting the graphs corresponding to adjacent letters appearing in W by identifying the very right vertex of the left letter with the very left vertex of the right letter. For example, the graph corresponding to the word $W = VD_3A$ is



Without risk of confusion, we shall use W to denote both a word over \mathcal{A} and its corresponding graph.

We now define $M(W)$ to be the kE -module with k -basis consisting of the vertices of W whose kE -module action is given as in Section 2.4, i.e., such that X_1 maps a vertex v to a vertex w if there is a single edge downwards to the left from v to w , and X_2 maps v to w if there is a double edge downwards to the right from v to w . Our main result regarding such modules is the following.

Theorem 2.22. *Let W be a word over \mathcal{A} satisfying the following conditions.*

- (i) W begins and ends with letters of the form D_n .
- (ii) If D_n appears in W , then $p \mid n$.
- (iii) W contains no strings of the form AA, VV, AV, VA or $D_m D_n$.

If the letters of the form D_n in W are (from left to right) D_{n_1}, \dots, D_{n_t} , then the corresponding module $M(W)$ has constant Jordan type $[3]^a [2]^b$ where

$$a = \sum_{i=1}^t (n_i - 1) - 1.$$

and $b = t + 1$.

PROOF. For convenience write $M = M(W)$. It is clear from inspecting the graph W that X_2 acts on M with Jordan type $[3]^a [2]^b$. In order to prove that M has constant Jordan type, it therefore suffices to prove that $\text{rank}(X_1 + \lambda X_2, M) = 2a + b$ and $\text{rank}((X_1 + \lambda X_2)^2, M) = a$ for all $\lambda \in k$.

The fact that $\text{rank}(X_1 + \lambda X_2, M) = 2a + b$ can be verified by noting that the map

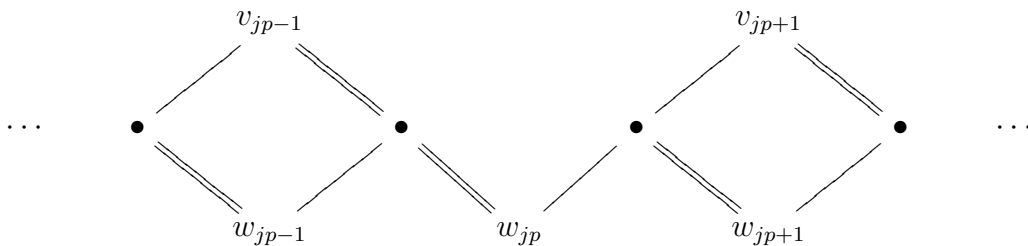
$$X_1 + \lambda X_2: M/\text{Rad}(M) \longrightarrow \text{Rad}(M)/\text{Rad}^2(M)$$

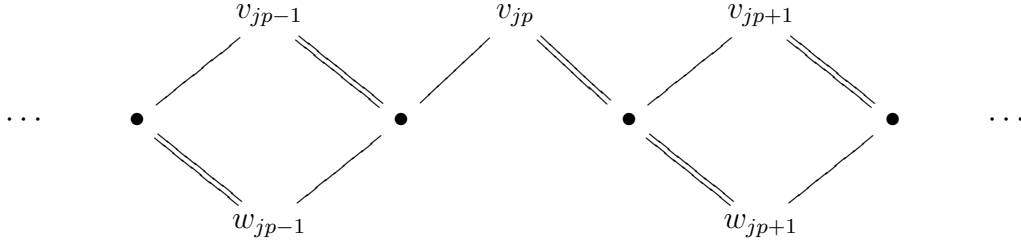
is injective, and the map

$$X_1 + \lambda X_2: \text{Rad}(M)/\text{Rad}^2(M) \longrightarrow \text{Rad}^2(M)$$

is surjective. (To see this, consider the proof of Proposition 4.14 and its dual.) In other words, $\text{rank}(X_1 + \lambda X_2, M)$ is the number of top vertices in W plus the number of bottom vertices in W .

To calculate $\text{rank}((X_1 + \lambda X_2)^2, M)$, we label the top and bottom vertices of W from left to right by v_1, v_2, \dots and w_1, w_2, \dots , respectively, increasing the index on v_i by one after passing a subgraph of the form V and increasing the index on w_i by one after passing a subgraph of the form A . This labeling scheme is illustrated in the following diagrams.





We now consider an element $\sum_i a_i v_i \in \text{Ker}((X_1 + \lambda X_2)^2, M)$, where $a_i \in k$. For convenience we set $a_i = 0$ when v_i does not exist. Examining the coefficients on w_i in $(X_1 + \lambda X_2)^2 \sum_j a_j v_j$, which must all equal zero, we obtain the recursion relations

$$\lambda^2 a_{i-1} + 2\lambda a_i + a_{i+1} = 0 \quad \text{if } w_i \text{ exists.} \quad (2.7)$$

Let C_1, \dots, C_s be the maximal subwords of W (from left to right) that do not contain the letter A so that $W = C_1 A C_2 A \dots A C_s$. For $1 \leq l \leq s$, let $j(l) \geq 0$ be the integer such that $v_{j(l)p+1}$ is the leftmost top vertex in C_l . We claim that any solution to the recurrence relations (2.7) is obtained by independently choosing the coefficients $a_{j(l)p+1}$. From this it will follow that the kernel of

$$(X_1 + \lambda X_2)^2: M/\text{Rad}(M) \rightarrow \text{Rad}^2(M)$$

has dimension s , showing that the rank of $(X_1 + \lambda X_2)^2$ on M is

$$(\text{number of top vertices in } W) - s = a$$

as required. In order to prove the claim, we will show for $1 \leq l \leq s$ that $a_{j(l)p+1}$ completely determines the coefficients a_i for $j(l)p + 1 \leq i \leq j(l+1)p$, proceeding by induction on l .

To prove the claim for $l = 1$, we first show that

$$a_i = (-\lambda)^{i-1} a_1 \quad \text{for all } 1 \leq i \leq j(2)p,$$

using induction on i . This clearly holds for $i = 1$. For $i > 1$, it follows from the $(i - 1)$ st relation and the inductive hypothesis that

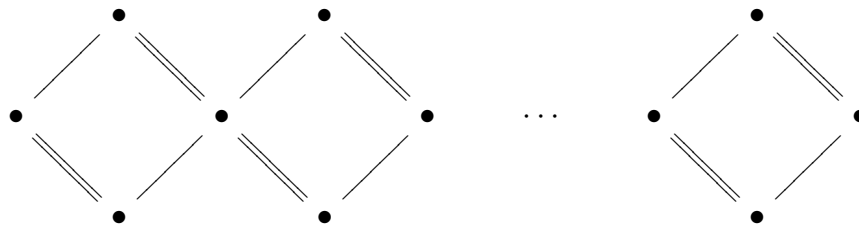
$$a_i = -2\lambda a_{i-1} - \lambda^2 a_{i-2} = -2\lambda(-\lambda)^{i-2}(i-1)a_1 - \lambda^2(-\lambda)^{i-3}(i-2)a_1 = (-\lambda)^{i-1} a_1$$

as required. (Note that if $p \mid i$, then $a_i = 0$ since k has characteristic p , hence this calculation is consistent with our convention that $a_i = 0$ if v_i does not exist.) This shows that a_1 determines a_i for all $1 \leq i \leq j(2)p$. In particular, we have $a_{j(2)p} = 0$. Note that $v_{j(2)}$ is the top vertex in the subgraph A directly following C_1 in W , hence there is no vertex $w_{j(2)p}$. Referring to the recurrence relations (2.7), because there

is no $j(2)p$ th relation, this shows that none of the coefficients a_i for $i \geq j(2)p + 1$ depends on a_1 . This establishes the claim for $l = 1$.

For $l > 1$, we assume by induction that $a_{j(l)p+1}$ does not depend on any of the previous a_i . The proof then follows by starting at the $(j(l)p + 1)$ st relation and using the same argument as in the case $l = 1$. \square

Remark 2.23. It follows from the above proof that if M is the kE -module having diagram



then M has constant Jordan type if and only if the number of middle vertices is divisible by p . This illustrates the necessity of condition (ii) in Theorem 2.22.

2.8. Some questions and conjectures

Having defined modules of constant Jordan type and examined some of their properties, it is now natural to consider what kind of information we would like to know about these modules, and which techniques will be effective in obtaining that information.

One might first be tempted to investigate modules of constant Jordan type in terms of their rank varieties. (See Appendix C.) The problem with this approach is that a module of constant Jordan type is either (a) projective, in which case its rank variety contains only the zero point, or (b) non-projective, in which case its restriction to any cyclic shifted subgroup is non-projective as a $k(\mathbb{Z}/p)$ -module, hence its rank variety is all of $\mathbb{A}^r(k)$. In other words, rank varieties are only capable of distinguishing two modules of constant Jordan type if one is projective and the other is non-projective.

Having established that rank varieties are too coarse to yield meaningful information about modules of constant Jordan type, one is left to search for finer invariants that *are* suitable for this purpose. Such a natural invariant would be the constant Jordan type itself. Having this observation in mind, Carlson, Friedlander and Pevtsova [16] posed the following general question: For which Jordan types $[p]^{a_p} \dots [1]^{a_1}$ does there exist a kE -module of constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$? This is sometimes called the *realisation problem* for modules of constant Jordan type.

We now record some conjectures that are related to the above question. The first is due to Suslin and appeared as Question 9.6 of [16].

Conjecture 2.24 (Suslin's conjecture). Let $r \geq 2$. If a kE -module has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ and $a_i \neq 0$ for some $2 \leq i \leq p-1$, then either $a_{i-1} \neq 0$ or $a_{i+1} \neq 0$.

The second conjecture we record was made by Jeremy Rickard at MSRI in 2008.

Conjecture 2.25 (Rickard's conjecture). Let $r \geq 2$. If a kE -module has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ and $a_i = 0$ for some $1 \leq i \leq p-1$, then $\sum_{j=i+1}^p a_j$ is divisible by p .

Note that neither of these conjectures implies the other. For example, Suslin's conjecture fails to rule out the realisation of the Jordan type $[3][2]$, whereas Rickard's conjecture fails to rule out the realisation of the Jordan type $[2]^2$. Nonetheless, both imply the following, which appeared as Conjecture 9.5 of [16].

Conjecture 2.26. If $p \geq 5$ and $r \geq 2$, then there does not exist a kE -module of stable constant Jordan type $[2]$.

While Suslin and Rickard's conjectures remain open, Conjecture 2.26 has been answered in the affirmative by the following, more general result of Benson [5].

Theorem 2.27. *If $r \geq 2$, then there does not exist a kE -module of stable constant Jordan type $[a]$ with $2 \leq a \leq p-2$.*

We remark that the proof of Theorem 2.27 is highly specialised, using exterior and symmetric powers of a module to derive contradictions modulo p on its dimension. In Chapter 3 we prove a slightly weaker version of this theorem using the theory of Chern classes for vector bundles on projective space. Benson's Theorem 6.1 related to Rickard's conjecture provides an alternative proof of Theorem 2.27.

The final conjecture we record appeared as Conjecture 9.7 of [16].

Conjecture 2.28. If $p \geq 5$ and a kE -module has stable constant Jordan type $[2][1]^n$, then $n \geq r-1$.

CHAPTER 3

Restrictions on Jordan types from vector bundles on projective space

The goal of this chapter is to study kE -modules of stable constant Jordan type $[a][b]$ with $1 \leq a, b \leq p-1$ using Chern classes for vector bundles on projective space. Our approach is motivated by a theorem of Benson [7]. We begin by recalling some basic concepts from coherent sheaves on projective space. The primary reference for this material is Hartshorne [30]. A valuable reference for the theory of vector bundles on projective space is Okonek, Schneider and Spindler [32].

3.1. Vector bundles on projective space

We observed in Section 2.2 that the elements $X_1, \dots, X_r \in J$ may be viewed as the coordinates of the affine r -space $\mathbb{A}^r(k)$. As in Appendix A, we consider the corresponding dual basis Y_1, \dots, Y_r , where each Y_i acts as a coordinate function on $\mathbb{A}^r(k)$ via the relation $Y_i(X_j) = \delta_{ij}$. Recall that $k[Y_1, \dots, Y_r]$ is an \mathbb{N} -graded ring, where the homogeneous elements of degree n are defined to be the k -linear combinations of monomials of degree n in Y_1, \dots, Y_r .

If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a \mathbb{Z} -graded $k[Y_1, \dots, Y_r]$ -module and \mathfrak{p} is a homogeneous prime ideal of $k[Y_1, \dots, Y_r]$, we denote by $M_{(\mathfrak{p})}$ the set of degree zero elements in the localisation $S^{-1}M$, where S is the set of homogeneous elements in $k[Y_1, \dots, Y_r]$ that do not lie in \mathfrak{p} . If n is any integer, we denote by $M(n)$ the module M with shifted grading given by $M(n)_i = M_{i+n}$ for all $i \in \mathbb{Z}$.

We now consider the projective space $\mathbb{P}^{r-1} = \text{Proj } k[Y_1, \dots, Y_r]$. In other words, \mathbb{P}^{r-1} is the set of homogeneous prime ideals in $k[Y_1, \dots, Y_r]$ that do not contain the ideal (Y_1, \dots, Y_r) . The topology on \mathbb{P}^{r-1} is defined by taking the closed sets to be those of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathbb{P}^{r-1} \mid \mathfrak{a} \subseteq \mathfrak{p}\},$$

where \mathfrak{a} is a homogeneous ideal of $k[Y_1, \dots, Y_r]$.

The structure sheaf $\mathcal{O} = \mathcal{O}_{\mathbb{P}^{r-1}}$ on \mathbb{P}^{r-1} is defined in the following way. For each open set $U \subseteq \mathbb{P}^{r-1}$, we let $\mathcal{O}(U)$ be the ring of functions

$$s: U \longrightarrow \prod_{\mathfrak{p} \in U} k[Y_1, \dots, Y_r]_{(\mathfrak{p})}$$

such that $s(\mathfrak{p}) \in k[Y_1, \dots, Y_r]_{(\mathfrak{p})}$ for all $\mathfrak{p} \in U$, and such that s is locally a homogeneous rational function of degree zero in Y_1, \dots, Y_r . Specifically, for each $\mathfrak{p} \in U$ there exists a neighbourhood V of \mathfrak{p} contained in U and homogeneous elements $a, f \in k[Y_1, \dots, Y_r]$ of the same degree such that, for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f \in k[Y_1, \dots, Y_r]_{(\mathfrak{q})}$.

Similarly, if M is any \mathbb{Z} -graded $k[Y_1, \dots, Y_r]$ -module, we construct the sheaf \tilde{M} of \mathcal{O} -modules on \mathbb{P}^{r-1} in the following way. For each open set $U \subseteq \mathbb{P}^{r-1}$, we let $\tilde{M}(U)$ be the $\mathcal{O}(U)$ -module whose elements are functions $s: U \rightarrow \prod_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$ such that, for all $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{(\mathfrak{p})}$ and there exists a neighbourhood V of \mathfrak{p} contained in U and homogeneous elements $m \in M$, $f \in k[Y_1, \dots, Y_r]$ of the same degree such that, for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f \in M_{(\mathfrak{q})}$. If M is finitely generated, then it follows from Proposition 5.11 (c) of [30, Chapter II] that \tilde{M} is a coherent sheaf on \mathbb{P}^{r-1} .

An important class of \mathcal{O} -modules constructed in the above way are the twists of the structure sheaf \mathcal{O} . Specifically, for $n \in \mathbb{Z}$, the n th twist of \mathcal{O} is the coherent sheaf $\mathcal{O}(n) = k[Y_1, \dots, Y_r](n)^\sim$. It is apparent from the above construction that sections of $\mathcal{O}(n)$ are given locally by homogenous rational functions of degree n in Y_1, \dots, Y_r . For any sheaf \mathcal{F} on \mathbb{P}^{r-1} , we define $\mathcal{F}(n)$ to be the tensor product sheaf $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(n)$. By Proposition 5.12 (b) of [30, Chapter II], we have $\mathcal{O}(m) \otimes_{\mathcal{O}} \mathcal{O}(n) \cong \mathcal{O}(m+n)$ for all $m, n \in \mathbb{Z}$.

A coherent sheaf \mathcal{F} on \mathbb{P}^{r-1} is *locally free* if there exists an open covering $\{U_i\}_{i \in I}$ of \mathbb{P}^{r-1} such that for each $i \in I$, the restriction $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}|_{U_i}$ -module, i.e., a direct sum of copies of $\mathcal{O}|_{U_i}$. Since \mathbb{P}^{r-1} is irreducible, we have $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$, hence the free rank of $\mathcal{F}|_{U_i}$ is independent of U_i . We therefore define the *rank* of \mathcal{F} to be the free rank of its restriction to any U_i . A *vector bundle* on \mathbb{P}^{r-1} is a locally free sheaf of finite rank. A vector bundle of rank one is called a *line bundle*.

Proposition 3.1. *The twisted sheaves $\mathcal{O}(n)$ for $n \in \mathbb{Z}$ are line bundles on \mathbb{P}^{r-1} .*

PROOF. Consider the open sets $U_i = D(Y_i) = \mathbb{P}^{r-1} \setminus V((Y_i))$ for $1 \leq i \leq r$. These form an affine open covering of \mathbb{P}^{r-1} since each homogeneous prime ideal in \mathbb{P}^{r-1} does not contain some Y_i . If s is a section in $\mathcal{O}(U_i)$ and $x \in U_i$, then $s(x)$ is a rational function of degree n in Y_1, \dots, Y_r . Letting $s/Y_i^n: U_i \rightarrow \prod_{x \in U_i} \mathcal{O}_x$ denote the function mapping each $x \in U_i$ to the rational function $s(x)/Y_i^n$ of degree zero, the assignment

$s \mapsto s/Y_i^n$ induces an isomorphism $\mathcal{O}(n)|_{U_i} \cong \mathcal{O}|_{U_i}$. This shows that $\mathcal{O}(n)$ is a vector bundle of rank one on \mathbb{P}^{r-1} . \square

Remark 3.2. The converse is also true, that is, every line bundle on \mathbb{P}^{r-1} is isomorphic to $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$. (See Corollary 6.17 of [30, Chapter II].)

Recall that the *residue field* of a point $x \in \mathbb{P}^{r-1}$ is the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$, where \mathfrak{m}_x is the unique maximal ideal of the local ring \mathcal{O}_x . If \mathcal{F} is any sheaf on \mathbb{P}^{r-1} , then the *fibre of \mathcal{F} at x* is defined to be the $k(x)$ -vector space $\mathcal{F}_x \otimes_{\mathcal{O}_x} k(x)$. The following proposition is a special case of Exercise 5.8 of [30, Chapter II]. It gives an essential criterion for a coherent sheaf on \mathbb{P}^{r-1} to be a vector bundle in terms of the respective dimensions of its fibres.

Proposition 3.3. *If \mathcal{F} is a coherent sheaf on \mathbb{P}^{r-1} , then \mathcal{F} is a vector bundle if and only if $\dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_x} k(x)$ is independent of the point $x \in \mathbb{P}^{r-1}$.*

An immediate consequence of Proposition 3.3 is that injectivity and surjectivity for maps of vector bundles can be detected on restriction to fibres.

Proposition 3.4. *If $\phi: \mathcal{F} \rightarrow \mathcal{F}'$ is a map of vector bundles on \mathbb{P}^{r-1} , then ϕ is a monomorphism (resp. epimorphism, isomorphism) if and only if the induced map of fibres*

$$\phi_x: \mathcal{F} \otimes_{\mathcal{O}_x} k(x) \longrightarrow \mathcal{F}' \otimes_{\mathcal{O}_x} k(x)$$

is a monomorphism (resp. epimorphism, isomorphism) for all $x \in \mathbb{P}^{r-1}$.

3.2. Trivial sheaves for kE -modules and the operator θ_M

Let M be a kE -module and set $d = \dim_k(M)$. We define the *trivial sheaf* associated to M to be the coherent sheaf $\widetilde{M} = M \otimes_k \mathcal{O}$ on \mathbb{P}^{r-1} whose sections on an open set $U \subseteq \mathbb{P}^{r-1}$ are $M \otimes_k \mathcal{O}(U)$.¹ Choosing a k -basis m_1, \dots, m_d of M , we may identify $\widetilde{M}(U)$ with the k -vector space $\mathcal{O}^{\oplus d}(U)$ via the relation

$$\sum_{i=1}^d \lambda_i m_i \otimes s \longmapsto (\lambda_1 s, \dots, \lambda_d s) \in \mathcal{O}^{\oplus d}(U).$$

It follows that $\widetilde{M} \cong \mathcal{O}^{\oplus d}$ as \mathcal{O} -modules. Passing through this isomorphism, we obtain

$$\widetilde{M}(n) = (M \otimes_k \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}(n) \cong M \otimes_k (\mathcal{O} \otimes_{\mathcal{O}} \mathcal{O}(n)) \cong M \otimes_k \mathcal{O}(n).$$

¹ The trivial sheaf associated to a kE -module is denoted by a wide tilde in order to distinguish it from the sheaf constructed from a graded $k[Y_1, \dots, Y_r]$ -module in Section 3.1.

If U is an open subset of \mathbb{P}^{r-1} and s is a section in $\mathcal{O}(n)(U)$, then for $1 \leq i \leq r$ we define $Y_i s$ to be the function $U \rightarrow \prod_{x \in U} \mathcal{O}(n+1)_x$ given by $x \mapsto Y_i s(x)$. Since $s(x)$ is a rational function of degree n in the coordinates Y_1, \dots, Y_r , we see that $Y_i s(x)$ is a rational function of degree $n+1$. The assignment $s \mapsto Y_i s$ induces a sheaf morphism $\mathcal{O}(n) \rightarrow \mathcal{O}(n+1)$. In fact, it follows from Theorem 5.1 of [30, Chapter III] that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(n), \mathcal{O}(n+1)) \cong H^0(\mathbb{P}^{r-1}, \mathcal{O}(1)) \cong \mathrm{span}_k(Y_1, \dots, Y_r).$$

Combining the above morphisms with the action of $X_1, \dots, X_r \in kE$ on M , we define $\theta_M: \widetilde{M}(n) \rightarrow \widetilde{M}(n+1)$ to be the operator given on each open set $U \subseteq \mathbb{P}^{r-1}$ by

$$\theta_{M,U}(m \otimes s) = \sum_{i=1}^r X_i m \otimes Y_i s \quad \text{for all } m \in M, s \in \mathcal{O}(n)(U).$$

The above construction is due to Friedlander and Pevtsova [25].

Note that the action of θ_M can be visualised in terms of matrices of linear forms. Specifically, if $\rho_M: kE \rightarrow \mathrm{Mat}_d(k)$ is the matrix representation affording M , then the matrix representing θ_M is given locally by

$$\sum_{i=1}^r Y_i \rho_M(X_i) \in \mathrm{Mat}_d(k[Y_1, \dots, Y_r]).$$

Now let $\phi: M \rightarrow N$ be a homomorphism of finitely generated kE -modules. We define the sheaf morphism $\tilde{\phi}: \widetilde{M} \rightarrow \widetilde{N}$ to be the map defined on each open subset $U \subseteq \mathbb{P}^{r-1}$ by

$$\tilde{\phi}_U(m \otimes s) = \phi(m) \otimes s \in \widetilde{N}(U) \quad \text{for all } m \in M, s \in \mathcal{O}(U).$$

Note for each $m \in M$ and $s \in \mathcal{O}(U)$ that we have

$$\begin{aligned} \tilde{\phi}_U(\theta_{M,U}(m \otimes s)) &= \tilde{\phi}_U\left(\sum_{i=1}^r X_i m \otimes Y_i s\right) = \sum_{i=1}^r \phi(X_i m) \otimes Y_i s = \sum_{i=1}^r X_i \phi(m) \otimes Y_i s \\ &= \theta_{N,U}(\tilde{\phi}_U(m \otimes s)), \end{aligned}$$

hence $\tilde{\phi} \circ \theta_M = \theta_N \circ \tilde{\phi}$.

Our immediate goal is to show that the kE -module structure of M is encoded in the fibres of the operator θ_M at points in \mathbb{P}^{r-1} . To illustrate this, we first recall the method of calculating the fibres of a sheaf \mathcal{F} on an arbitrary scheme (X, \mathcal{O}_X) . Since the fibre of \mathcal{F} at a point $x \in X$ is equal to the fibre of its restriction to any affine open set containing x , we may assume that $X = \mathrm{Spec} A$ for some commutative ring A . If \mathfrak{p} is any prime ideal of A , then $k(\mathfrak{p})$ is an A -module under the composition of ring homomorphisms

$$A \longrightarrow A_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$$

and we have

$$\mathcal{F}_{\mathfrak{p}} \otimes_{\mathcal{O}_{x,\mathfrak{p}}} k(\mathfrak{p}) \cong \Gamma(\mathcal{F}) \otimes_A k(\mathfrak{p}).$$

Applying this to the sheaf $\widetilde{M}(n)$ on \mathbb{P}^{r-1} , recall that if $x \in \mathbb{P}^{r-1}$ then there exists a coordinate function Y_i such that $x \in U_i = \mathbb{P}^{r-1} \setminus V((Y_i))$. Because $\mathcal{O}(n)|_{U_i} \cong \mathcal{O}|_{U_i}$ by the proof of Proposition 3.1, it follows that

$$\widetilde{M}(n)_x \otimes_{\mathcal{O}_x} k(x) \cong M \otimes_k \mathcal{O}(U_i) \otimes_{\mathcal{O}(U_i)} k(x) \cong M \otimes_k k(x).$$

To calculate the fibre of the morphism $\theta_M: \widetilde{M}(n) \rightarrow \widetilde{M}(n+1)$ at $x \in \mathbb{P}^{r-1}$, note that $\mathcal{O}(U_i)$ is isomorphic to the degree zero elements in the localised ring $k[Y_1, \dots, Y_r]_{Y_i}$ by Proposition 2.5 of [30, Chapter II], and this in turn is equal to $k[Y_1 Y_i^{-1}, \dots, Y_r Y_i^{-1}]$. We extend k to $k(x)$ by forming the fibred product

$$\mathbb{P}_{k(x)}^{r-1} = \mathbb{P}^{r-1} \times_{\text{Spec } k} \text{Spec } k(x) \cong \text{Proj } k(x)[Y_1, \dots, Y_r].$$

If $y \in \mathbb{P}_{k(x)}^{r-1}$ is the generic point of x (see Section A.2), then x is the image of y under the projection $\mathbb{P}_{k(x)}^{r-1} \rightarrow \mathbb{P}^{r-1}$, and the fibre of θ_M at x is equal to the fibre of θ_M at y . To calculate the latter, it suffices to restrict θ_M to the affine open set

$$U_{i,k(x)} = U_i \times_{\text{Spec } k} \text{Spec } k(x) \cong \text{Spec } k(x)[Y_1 Y_i^{-1}, \dots, Y_r Y_i^{-1}]$$

containing y . Because y is closed in $\mathbb{P}_{k(x)}^{r-1}$, it corresponds to a point

$$\bar{\alpha} = [\lambda_1 : \dots : \lambda_r]$$

in the projective variety $\mathbb{P}^{r-1}(k(x))$ whose i th homogeneous coordinate is non-zero. The $k(x)[Y_1 Y_i^{-1}, \dots, Y_r Y_i^{-1}]$ -module structure on $k(x)$ is then given via the specialisation

$$\begin{aligned} k(x)[Y_1 Y_i^{-1}, \dots, Y_r Y_i^{-1}] &\longrightarrow k(x) \\ f &\longmapsto f(\alpha), \end{aligned}$$

where $\alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k(x))$ is any point lying above $\bar{\alpha}$. It follows that the fibre of θ_M at y maps the element

$$m \otimes 1 \otimes 1 \in M \otimes_k \mathcal{O}(U_{i,k(x)}) \otimes_{\mathcal{O}(U_{i,k(x)})} k(x)$$

to

$$\sum_{j=1}^r X_j m \otimes Y_j \otimes 1 = \sum_{j=1}^r X_j m \otimes 1 \otimes \lambda_j \lambda_i^{-1} \in M \otimes_k k(x).$$

This establishes the following, which appeared implicitly in [25].

Proposition 3.5. *Let M be a finitely generated kE -module and $x \in \mathbb{P}^{r-1}$. Then the fibre of $\widetilde{M}(n)$ at x is isomorphic to the extension $M \otimes_k k(x)$ as a $k(x)$ -vector space. If $\bar{\alpha} = [\lambda_1 : \dots : \lambda_r] \in \mathbb{P}^{r-1}(k(x))$ corresponds to the generic point of x in $\mathbb{P}_{k(x)}^{r-1}$ and*

$\alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k(x))$ is any point lying above $\bar{\alpha}$, then up to a scalar in $k(x)$, the fibre of $\theta_M: \widetilde{M}(n) \rightarrow \widetilde{M}(n+1)$ at x is given by the action of $X_\alpha = \sum_j X_j \otimes \lambda_j$ on $M \otimes_k k(x)$ as a $(kE \otimes_k k(x))$ -module.

3.3. The functors $\mathcal{F}_{i,j}$ and \mathcal{F}_i

Let M be a finitely generated kE -module. For $1 \leq i \leq p$ and $0 \leq j < i$, Benson and Pevtsova [9] have constructed the subquotient

$$\mathcal{F}_{i,j}(M) = \frac{\text{Ker } \theta_M^{j+1} \cap \text{Im } \theta_M^{i-j-1}}{(\text{Ker } \theta_M^{j+1} \cap \text{Im } \theta_M^{i-j}) + (\text{Ker } \theta_M^j \cap \text{Im } \theta_M^{i-j-1})}$$

of \widetilde{M} where, for each exponent l in the above expression, $\text{Ker } \theta_M^l$ denotes the kernel of the sheaf morphism $\theta_M: \widetilde{M} \rightarrow \widetilde{M}(l)$, and $\text{Im } \theta_M^l$ denotes the image of the sheaf morphism $\theta_M: \widetilde{M}(-l) \rightarrow \widetilde{M}$. In the case $j = 0$, the authors of [9] also define

$$\mathcal{F}_i(M) = \mathcal{F}_{i,0}(M) = \frac{\text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}}{\text{Ker } \theta_M \cap \text{Im } \theta_M^i}.$$

Note that $\mathcal{F}_{i,j}(M)$ is a coherent sheaf of \mathcal{O} -modules on \mathbb{P}^{r-1} by Proposition 5.7 of [30, Chapter II]. Moreover, if $\phi: M \rightarrow N$ is any kE -module homomorphism, then there is an obvious sheaf morphism

$$\mathcal{F}_{i,j}(\phi): \mathcal{F}_{i,j}(M) \longrightarrow \mathcal{F}_{i,j}(N)$$

factoring through $\tilde{\phi}: \widetilde{M} \rightarrow \widetilde{N}$. This map is well defined since $\tilde{\phi} \circ \theta_M = \theta_N \circ \tilde{\phi}$. (See Section 3.2 for further details.) It follows for each $0 \leq j < i \leq p$ that $\mathcal{F}_{i,j}$ is a functor from the category $\mathbf{mod}(kE)$ of finitely generated kE -modules to the category $\mathbf{Coh}(\mathbb{P}^{r-1})$ of coherent sheaves on \mathbb{P}^{r-1} . The following proposition, which appeared as Proposition 2.1 of [9], shows that the functors \mathcal{F}_i map modules of constant Jordan type to vector bundles.

Proposition 3.6. *A kE -module M has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ if and only if $\mathcal{F}_i(M)$ is a vector bundle of rank a_i on \mathbb{P}^{r-1} for all $1 \leq i \leq p$. Hence the assignment $M \mapsto \mathcal{F}_i(M)$ defines a functor from the category $\mathbf{cJt}(kE)$ of kE -modules of constant Jordan type (see Section 2.6) to the category $\mathbf{Vec}(\mathbb{P}^{r-1})$ of vector bundles on \mathbb{P}^{r-1} .*

SKETCH OF PROOF. Recall from Proposition 3.5 that the fibre of θ_M at a point $x \in \mathbb{P}^{r-1}$ is given by the action of $X_\alpha \in kE \otimes_k k(x)$ on $M \otimes_k k(x)$, where

$$\bar{\alpha} = [\lambda_1 : \dots : \lambda_r] \in \mathbb{P}^{r-1}(k(x))$$

corresponds to the generic point of x in the scheme $\mathbb{P}_{k(x)}^{r-1}$. The fibre of $\mathcal{F}_i(M)$ at x is then equal to

$$\frac{\text{Ker}(X_\alpha, M \otimes_k k(x)) \cap \text{Im}(X_\alpha^{i-1}, M \otimes_k k(x))}{\text{Ker}(X_\alpha, M \otimes_k k(x)) \cap \text{Im}(X_\alpha^i, M \otimes_k k(x))}.$$

If M has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$, it follows from Proposition 2.10 that the rank of X_α^s on $M \otimes_k k(x)$ is independent of x for each $s \geq 0$. This means that the dimension of the above expression is independent of $x \in \mathbb{P}^{r-1}$ and equal to a_i . By Proposition 3.3, this shows that $\mathcal{F}_i(M)$ is a vector bundle of rank a_i on \mathbb{P}^{r-1} . The converse follows by a similar argument. \square

Our next proposition appeared as Lemma 2.2 of [9].

Proposition 3.7. *If M is a finitely generated kE -module, then \widetilde{M} has a filtration with filtered quotients $\mathcal{F}_{i,j}(M)$ for all $0 \leq j < i \leq p$.*

PROOF. Consider the kernel and image filtrations of \widetilde{M} :

$$\begin{aligned} 0 &= \text{Ker } \theta_M^0 \subseteq \text{Ker } \theta_M \subseteq \dots \subseteq \text{Ker } \theta_M^p = \widetilde{M}, \\ 0 &= \text{Im } \theta_M^p \subseteq \text{Im } \theta_M^{p-1} \subseteq \dots \subseteq \text{Im } \theta_M^0 = \widetilde{M}. \end{aligned}$$

For ease of notation, set $\mathcal{K}_j = \text{Ker } \theta_M^j$ and $\mathcal{I}_l = \text{Im } \theta_M^{p-l}$. Mimicking the proof of the Jordan-Hölder theorem, we refine the kernel filtration by

$$\mathcal{K}_j \subseteq (\mathcal{K}_{j+1} \cap \mathcal{I}_1) + \mathcal{K}_j \subseteq \dots \subseteq (\mathcal{K}_{j+1} \cap \mathcal{I}_{p-1}) + \mathcal{K}_j \subseteq \mathcal{K}_{j+1}.$$

If \mathcal{F} , \mathcal{G} and \mathcal{H} are subsheaves of \widetilde{M} with $\mathcal{G} \subseteq \mathcal{F}$, then by the second isomorphism theorem and the modular law we have

$$\frac{\mathcal{F} + \mathcal{H}}{\mathcal{G} + \mathcal{H}} = \frac{\mathcal{F} + (\mathcal{G} + \mathcal{H})}{\mathcal{G} + \mathcal{H}} \cong \frac{\mathcal{F}}{\mathcal{F} \cap (\mathcal{G} + \mathcal{H})} = \frac{\mathcal{F}}{\mathcal{G} + (\mathcal{F} \cap \mathcal{H})}.$$

Applying this to the above refined filtration, we obtain

$$\frac{(\mathcal{K}_{j+1} \cap \mathcal{I}_{l+1}) + \mathcal{K}_j}{(\mathcal{K}_{j+1} \cap \mathcal{I}_l) + \mathcal{K}_j} \cong \frac{\mathcal{K}_{j+1} \cap \mathcal{I}_{l+1}}{(\mathcal{K}_{j+1} \cap \mathcal{I}_l) + (\mathcal{K}_j \cap \mathcal{I}_{l+1})}.$$

Setting $l = p - i + j$, the right hand expression is equal to $\mathcal{F}_{i,j}(M)$. \square

The following proposition appeared as Lemma 2.3 of [9]. In particular, it shows for each kE -module M that $\mathcal{F}_{i,j}(M) \cong \mathcal{F}_i(M)(j)$.

Proposition 3.8. *There is a natural isomorphism of functors $\mathcal{F}_{i,j} \cong \mathcal{F}_i(-)(j)$ for all $0 \leq j < i \leq p$.*

PROOF. The map $\theta_M: \widetilde{M} \rightarrow \widetilde{M}(1)$ induces an isomorphism

$$\mathcal{F}_{i,j}(M) \xrightarrow{\cong} \mathcal{F}_{i,j-1}(M)(1).$$

We obtain $\mathcal{F}_{i,j}(M) \cong \mathcal{F}_i(M)(j)$ by induction on j , noting that $\mathcal{F}_i(M) = \mathcal{F}_{i,0}(M)$. Naturality can be checked at each step of the induction. \square

The final result of this section describes how the functors \mathcal{F}_i behave with respect to taking duals in $\mathbf{mod}(kE)$. Recall that there is also a concept of dualisation for vector bundles. Specifically, for a vector bundle \mathcal{F} on \mathbb{P}^{r-1} , we define $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$. Here, $\mathcal{H}om_{\mathcal{O}}$ denotes the operation of taking the *sheaf hom* of two sheaves on \mathbb{P}^{r-1} . (See Exercise 1.15 of [30, Chapter II].) It is routine to verify that \mathcal{F}^\vee is also a vector bundle on \mathbb{P}^{r-1} , and that if \mathcal{G} is any sheaf on \mathbb{P}^{r-1} , then there is an isomorphism

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \cong \mathcal{F}^\vee \otimes_{\mathcal{O}} \mathcal{G}.$$

The following appeared as Theorem 3.6 of [9].

Proposition 3.9. *If M is a kE -module, then*

$$\mathcal{F}_i(M^*) \cong \mathcal{F}_i(M)^\vee(-i+1)$$

for all $1 \leq i \leq p$.

3.4. Chern classes for vector bundles on \mathbb{P}^{r-1}

In this section we recall some basic notions from the theory of Chern classes for vector bundles on \mathbb{P}^{r-1} . The standard reference for this material is Fulton [27]. We conclude by presenting a lemma due to Benson and Pevtsova concerning congruences modulo p for Chern classes that will be essential in the following sections.

Recall that the *Chow ring* of \mathbb{P}^{r-1} is the graded commutative ring

$$A^*(\mathbb{P}^{r-1}) = \bigoplus_{i \geq 0} A^i(\mathbb{P}^{r-1}),$$

where $A^i(\mathbb{P}^{r-1})$ is the free abelian group generated by the closed irreducible subvarieties of codimension i in \mathbb{P}^{r-1} , modulo rational equivalence. By the discussion at the beginning of [27, Section 8.4], we have $A^*(\mathbb{P}^{r-1}) \cong \mathbb{Z}[h]/(h^r)$. If \mathcal{F} is a vector bundle on \mathbb{P}^{r-1} , then there exists a well defined polynomial class

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F})h + c_2(\mathcal{F})h^2 + \cdots + c_{r-1}(\mathcal{F})h^{r-1} \in A^*(\mathbb{P}^{r-1})$$

characterised by the following properties.

- (1) $c_i(\mathcal{F}) = 0$ for all $i \geq \text{rank}(\mathcal{F})$.
- (2) If $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is a short exact sequence of vector bundles on \mathbb{P}^{r-1} , then $c(\mathcal{F}_2) = c(\mathcal{F}_1)c(\mathcal{F}_3)$.
- (3) $c(\mathcal{O}(n)) = 1 + nh$ for all $n \in \mathbb{Z}$.

The equivalence class $c(\mathcal{F}) \in A^*(\mathbb{P}^{r-1})$ is called the *Chern polynomial* of \mathcal{F} , and the integers $c_i(\mathcal{F})$ are called the *Chern numbers* of \mathcal{F} . We record the following formula for the Chern numbers of twists, which follows from Example 3.2.2 of [27].

Lemma 3.10. *If \mathcal{F} is a vector bundle on \mathbb{P}^{r-1} , then for all $n \in \mathbb{Z}$, the i th Chern number of $\mathcal{F}(n)$ is given by*

$$c_i(\mathcal{F}(n)) = \sum_{j=0}^i n^j \binom{\text{rank}(\mathcal{F}) - i + j}{j} c_{i-j}(\mathcal{F}).$$

The following result, which appeared as Lemma 6.2 of Benson and Pevtsova [9], may be thought of as a vector bundle analog of Fermat's little theorem. Used in conjunction with Proposition 3.7, it will provide an essential tool for obtaining congruences modulo p on the Chern numbers of the trivial sheaf \widetilde{M} , where M is a kE -module of constant Jordan type.

Lemma 3.11. *If \mathcal{F} is a vector bundle on \mathbb{P}^{r-1} , then*

$$c(\mathcal{F})c(\mathcal{F}(1)) \cdots c(\mathcal{F}(p-1)) \equiv 1 - \text{rank}(\mathcal{F})h^{p-1} \pmod{(p, h^p)}.$$

3.5. Benson's theorem on small Jordan types

We now present the main result of Benson [7], along with its proof. It can be interpreted as saying the following: Given a stable Jordan type containing some block of length greater than one, there always exist p and r sufficiently large such that there does not exist a kE -module of constant Jordan type realising the stable Jordan type. The goal throughout the remainder of the chapter will be to tighten the bounds on p and r provided by the following theorem in a few special cases.

Theorem 3.12. *If a kE -module has constant Jordan type $[p]^{a_p} \cdots [1]^{a_1}$ and*

$$\sum_{i=1}^{p-1} i \cdot a_i \leq \min(r-1, p-2),$$

then $a_i = 0$ for all $2 \leq i \leq p-1$.

PROOF. Let M be a kE -module of constant Jordan type satisfying the hypothesis of the theorem and consider the trivial sheaf \widetilde{M} associated to M on \mathbb{P}^{r-1} , which is isomorphic to a direct sum of copies of the structure sheaf \mathcal{O} . By Proposition 3.7, \widetilde{M} has a filtration with filtered quotients $\mathcal{F}_i(M)(j)$ for all $0 \leq j < i \leq p$. Since Chern polynomials are multiplicative over short exact sequences, it follows by Lemma 3.11

that

$$1 = c(\widetilde{M}) = \prod_{0 \leq j < i \leq p} c(\mathcal{F}_i(M)(j)) \equiv \prod_{0 \leq j < i \leq p-1} c(\mathcal{F}_i(M)(j)) \pmod{(p, h^{p-1}, h^r)}.$$

Letting $a = \sum_{i=1}^{p-1} i \cdot a_i$, the polynomial on the right hand side has degree a , because each $\mathcal{F}_i(M)$ has rank a_i . Since $a \leq r - 1$ and $a \leq p - 2$, the above congruence can be viewed as an equality in $\mathbb{F}_p[h]$. The only units in $\mathbb{F}_p[h]$ are the constant polynomials, so all of the above factors must be equal to $1 \in \mathbb{F}_p[h]$. In particular, for $i \geq 2$, this is true of $c(\mathcal{F}_i(M))$ and $c(\mathcal{F}_i(M)(1))$. For such i , this implies that p divides $c_1(\mathcal{F}_i(M))$ and $c_1(\mathcal{F}_i(M)(1))$. We have

$$c_1(\mathcal{F}_i(M)(1)) = c_1(\mathcal{F}_i(M)) + \text{rank}(\mathcal{F}_i(M))$$

by Lemma 3.10, so p also divides the rank of $\mathcal{F}_i(M)$. Now observe that

$$\text{rank}(\mathcal{F}_i(M)) = a_i \leq a < p$$

by Proposition 3.6. It follows that $\text{rank}(\mathcal{F}_i(M)) = 0$ so that $a_i = 0$ as required. \square

3.6. Restrictions on the stable Jordan type $[a][b]$, $a \neq b$

Although Theorem 3.12 does provide restrictions on p and r for which there exists a kE -module of constant Jordan type realising a given stable Jordan type having some block of length greater than one, it is natural to ask whether or not these restrictions are sharp. As we shall see, this question has a negative answer in general.

The technique we use in obtaining sharper bounds is similar to that used in proving Theorem 3.12, but rather than focusing only on the first Chern numbers of the first twists of the bundles $\mathcal{F}_i(M)$, we exploit even higher Chern numbers of these bundles, and do so simultaneously to solve congruences modulo p on the first few Chern numbers of the trivial sheaf \widetilde{M} . Unfortunately, the calculations required in this process become increasingly difficult for larger stable Jordan types, and we shall restrict our attention to kE -modules whose stable constant Jordan types contain only two blocks. To illustrate the set-up, we first use our technique to prove a slightly weaker version of Theorem 2.27.

Theorem 3.13. *If $p \geq 5$ and $r \geq 3$, then there does not exist a kE -module of stable constant Jordan type $[a]$ with $2 \leq a \leq p - 2$.*

PROOF. Suppose M has stable constant Jordan type $[a]$. Let $\alpha = c_1(\mathcal{F}_a(M))$ be the first Chern number of $\mathcal{F}_a(M)$, which is a rank one vector bundle by Proposition 3.6. It follows by the properties of Chern polynomials that all higher Chern numbers

of $\mathcal{F}_a(M)$ are zero. Using Proposition 3.7 and Lemmas 3.10 and 3.11, we have

$$1 = c(\widetilde{M}) \equiv (1 + \alpha h)(1 + (\alpha + 1)h) \cdots (1 + (\alpha + a - 1)h) \pmod{(p, h^{p-1}, h^r)}.$$

Since $p \geq 5$ and $r \geq 3$, the coefficients on h and h^2 in this polynomial must be divisible by p . For convenience, we normalise by setting $\alpha' = \alpha + \frac{1}{2}(a - 1)$. The coefficient on h is then $a\alpha'$ so that $\alpha' \equiv 0 \pmod{p}$. One then calculates the coefficient on h^2 to be

$$\sum_{-\frac{a-1}{2} \leq n < m \leq \frac{a-1}{2}} (\alpha' + n)(\alpha' + m) = -\frac{1}{24}a(a+1)(a-1)$$

after substituting $\alpha' \equiv 0$. The latter expression must also be congruent to 0 \pmod{p} , contradicting our assumption that $2 \leq a \leq p - 2$. \square

We now use the above technique to obtain restrictions on the values of a and b for which there exists a kE -module of stable constant Jordan type $[a][b]$ with $a \neq b$, provided p and r are sufficiently large. The following is the main theorem of this section.

Theorem 3.14. *Suppose $p \geq 5$ and $r \geq 4$. If a kE -module has stable constant Jordan type $[a][b]$ with $a \neq b$, then one of the following holds.*

- (i) $a = p - b$.
- (ii) $a = p - b \pm 1$.
- (iii) $a^2 + b^2 - ab - 1 \equiv 0 \pmod{p}$.

PROOF. Suppose M has stable constant Jordan type $[a][b]$ and that $a \neq p - b$. Since $\mathcal{F}_a(M)$ and $\mathcal{F}_b(M)$ are vector bundles of rank one, their Chern polynomials are of the form $c(\mathcal{F}_a(M)) = 1 + \alpha h$ and $c(\mathcal{F}_b(M)) = 1 + \beta h$, respectively. As in the proof of Theorem 3.13, we have

$$1 = c(\widetilde{M}) \equiv \prod_{n=0}^{a-1} (1 + (\alpha + n)h) \prod_{m=0}^{b-1} (1 + (\beta + m)h) \pmod{(p, h^{p-1}, h^r)}.$$

Because $p \geq 5$ and $r \geq 4$, the coefficients on h , h^2 and h^3 must all be divisible by p . For convenience, we normalise by setting $\alpha' = \alpha + \frac{1}{2}(a - 1)$ and $\beta' = \beta + \frac{1}{2}(b - 1)$. The coefficient on h is $a\alpha' + b\beta'$ so that

$$\beta' \equiv -\frac{a}{b}\alpha' \pmod{p}. \quad (3.1)$$

The coefficient on h^2 is

$$\begin{aligned} & \sum_{-\frac{a-1}{2} \leq n < m \leq \frac{a-1}{2}} (\alpha' + n)(\alpha' + m) + \sum_{\substack{-\frac{a-1}{2} \leq n \leq \frac{a-1}{2} \\ -\frac{b-1}{2} \leq m \leq \frac{b-1}{2}}} (\alpha' + n)(\beta' + m) \\ & + \sum_{-\frac{b-1}{2} \leq n < m \leq \frac{b-1}{2}} (\beta' + n)(\beta' + m) = -\frac{a(a+b)}{2b} \alpha'^2 - \frac{(a+b)(a^2 + b^2 - ab - 1)}{24} \end{aligned}$$

after substituting (3.1). Since $a \neq p - b$, this implies

$$\alpha'^2 \equiv -\frac{b(a^2 + b^2 - ab - 1)}{12a} \pmod{p}. \quad (3.2)$$

The coefficient on h^3 is

$$\begin{aligned} & \sum_{-\frac{a-1}{2} \leq n < m < l \leq \frac{a-1}{2}} (\alpha' + n)(\alpha' + m)(\alpha' + l) + \sum_{-\frac{b-1}{2} \leq n < m < l \leq \frac{b-1}{2}} (\beta' + n)(\beta' + m)(\beta' + l) \\ & + \sum_{\substack{-\frac{a-1}{2} \leq n < m \leq \frac{a-1}{2} \\ -\frac{b-1}{2} \leq l \leq \frac{b-1}{2}}} (\alpha' + n)(\alpha' + m)(\beta' + l) + \sum_{\substack{-\frac{b-1}{2} \leq n < m \leq \frac{b-1}{2} \\ -\frac{a-1}{2} \leq l \leq \frac{a-1}{2}}} (\alpha' + l)(\beta' + n)(\beta' + m) \\ & = -\frac{a(a+b)(a-b)}{3b^2} \alpha'^3 + \frac{a(a+b)(a-b)}{12} \alpha' \end{aligned}$$

after substituting (3.1). Since $a \neq b$ and $a \neq p - b$, we have

$$\alpha'^3 - \frac{1}{4}b^2\alpha' \equiv 0 \pmod{p}$$

so that $\alpha' \equiv 0$ or $\pm \frac{1}{2}b \pmod{p}$. Substituting $\alpha' \equiv 0$ into (3.2) yields

$$a^2 + b^2 - ab - 1 \equiv 0 \pmod{p},$$

and substituting $\alpha' \equiv \pm \frac{1}{2}b$ into (3.2) yields

$$a^2 + b^2 + 2ab - 1 \equiv 0 \pmod{p}.$$

The last congruence has solutions $b \equiv -a \pm 1 \pmod{p}$, completing the proof. \square

Corollary 3.15. *Let $p \geq 5$ and $r \geq 4$. If a kE -module has stable constant Jordan type $[a][1]$ with $2 \leq a \leq p - 1$, then $a = p - 1$ or $a = p - 2$.*

PROOF. Conditions (i) and (ii) of Theorem 3.14 imply the result. If condition (iii) holds, then $a^2 - a \equiv 0 \pmod{p}$, forcing $a = 1$ or $a = p$, a contradiction. \square

Corollary 3.16. *If $p \geq 5$, $r \geq 4$ and $2 \leq a \leq p - 3$, then there does not exist a kE -module of stable constant Jordan type $[a][1]$.*

Remark 3.17. Note that Theorem 3.12 guarantees the stable Jordan type $[a][1]$ is not realised provided $r \geq a + 2$ and $p \geq a + 3$, whereas Corollary 3.16 allows us

to deduce the same statement whenever $r \geq 4$. Hence our technique provides new information in the cases where $4 \leq r \leq a + 1$.

Example 3.18. If $p \geq 7$ and $r \geq 4$, then there does not exist a kE -module of stable constant Jordan type $[3][1]$.

The above restriction technique can be applied to many other stable Jordan types $[a][b]$. For example we have the following.

Corollary 3.19. *If $p \geq 7$ and $r \geq 4$, then there does not exist a kE -module of stable constant Jordan type $[3][2]$.*

PROOF. Since $p \geq 7$, conditions (i) and (ii) of Theorem 3.14 cannot hold. Hence if such a module exists, then condition (iii) must hold so that $6 \equiv 0 \pmod{p}$. It follows that $p = 2$ or 3 , a contradiction. \square

We end this section by calculating which stable Jordan types $[a][b]$ would be excluded from being realised by Suslin's Conjecture 2.24 in light of the restrictions given by Theorem 3.14.

Proposition 3.20. *Suppose $p \geq 5$, $r \geq 4$ and that Conjecture 2.24 holds. If a kE -module has stable constant Jordan type $[a][b]$ with $a \neq b$, then this stable Jordan type is either $[p-1][1]$ or $\left[\frac{p+1}{2}\right] \left[\frac{p-1}{2}\right]$.*

PROOF. Suppose there exists a module of stable constant Jordan type $[a][b]$ and that this stable Jordan type is not $[p-1][1]$ or $\left[\frac{p+1}{2}\right] \left[\frac{p-1}{2}\right]$. By Conjecture 2.24, we must have $b = a \pm 1$. Without loss of generality, assume $2 \leq a \leq p-2$. Since $p \geq 5$, conditions (i) and (ii) of Theorem 3.14 cannot hold. So assume condition (iii) holds, that is,

$$a^2 + b^2 - ab - 1 \equiv 0 \pmod{p}.$$

Then

$$0 \equiv a^2 + (a \pm 1)^2 - a(a \pm 1) - 1 = a^2 \pm a$$

so that $a \equiv \pm 1 \pmod{p}$, a contradiction. \square

3.7. Restrictions on the stable Jordan type $[a]^2$

We now use the technique from Section 3.6 to place restrictions on the value of a for which there exists a kE -module of stable constant Jordan type $[a]^2$ with $2 \leq a \leq p-2$ provided p and r are sufficiently large.

Theorem 3.21. *Suppose $p \geq 7$ and $r \geq 5$. If a kE -module has stable constant Jordan type $[a]^2$ with $2 \leq a \leq p - 2$, then $a = \frac{p-1}{2}$ or $a = \frac{p+1}{2}$.*

PROOF. Suppose M has stable constant Jordan type $[a]^2$. Letting $u = c_1(\mathcal{F}_a(M))$ and $v = c_2(\mathcal{F}_a(M))$, we have

$$c_1(\mathcal{F}_a(M)(n)) = u + 2n \quad \text{and} \quad c_2(\mathcal{F}_a(M)(n)) = v + un + n^2$$

by Lemma 3.10. As in the proof of Theorem 3.14, we then obtain

$$1 = c(\widetilde{M}) \equiv \prod_{n=0}^{a-1} (1 + (u + 2n)h + (v + un + n^2)h^2) \pmod{(p, h^{p-1})}.$$

Since $p \geq 7$ and $r \geq 5$, the coefficients on h , h^2 , h^3 and h^4 must all be divisible by p . The coefficient on h is $au + a(a - 1)$, so $u \equiv -(a - 1) \pmod{p}$. The coefficient on h^2 is then

$$\sum_{n=0}^{a-1} (v + un + n^2) + \sum_{0 \leq n < m \leq a-1} (u + 2n)(u + 2m) = av - \frac{1}{6}a(a - 1)(2a - 1)$$

after substituting $u \equiv -(a - 1)$. Hence $v \equiv \frac{1}{6}(a - 1)(2a - 1)$. The coefficient on h^3 is

$$\sum_{\substack{0 \leq n, m \leq a-1 \\ n \neq m}} (v + un + n^2)(u + 2m) + \sum_{0 \leq n < m < l \leq a-1} (u + 2n)(u + 2m)(u + 2l).$$

Unfortunately, after substituting $u \equiv -(a - 1)$, this expression is identically zero in v . Hence we are forced to consider the coefficient on h^4 , which is calculated using

$$\begin{aligned} & \sum_{0 \leq n < m \leq a-1} (v + un + n^2)(v + um + m^2) + \sum_{\substack{0 \leq n, m, l \leq a-1 \\ m \neq n \neq l \\ m < l}} (v + un + n^2)(u + 2m)(u + 2l) \\ & + \sum_{0 \leq n < m < l < k \leq a-1} (u + 2n)(u + 2m)(u + 2l)(u + 2k). \end{aligned}$$

Substituting $u \equiv -(a - 1)$ and $v \equiv \frac{1}{6}(a - 1)(2a - 1)$, this yields

$$\frac{1}{360}a(a + 1)(a - 1)(2a + 1)(2a - 1),$$

which is congruent to 0 (mod p). Since $2 \leq a \leq p - 2$, we must have $2a + 1 \equiv 0$ or $2a - 1 \equiv 0$ so that $a = \frac{p-1}{2}$ or $a = \frac{p+1}{2}$. \square

Corollary 3.22. *If $p \geq 11$ and $r \geq 5$, then there does not exist a kE -module of stable constant Jordan type $[3]^2$.*

3.8. Non-realisation of the stable Jordan type $[3][2][1]$ in certain cases

The final result of this chapter uses the above techniques to place restrictions on the values of p and r for which there exists a kE -module of stable constant Jordan type $[3][2][1]$.

Theorem 3.23. *If $p \geq 11$ and $r \geq 6$ or $p \geq 13$ and $r \geq 5$, then there does not exist a kE -module of stable constant Jordan type $[3][2][1]$.*

PROOF. Using the same technique as before, we assume that M is a module of stable constant Jordan type $[3][2][1]$ and write

$$(1 + \alpha h)(1 + \beta h)(1 + (\beta + 1)h)(1 + \gamma h)(1 + (\gamma + 1)h)(1 + (\gamma + 2)h) \equiv 1 \pmod{(p, h^{p-1})}$$

where $\alpha = c_1(\mathcal{F}_1(M))$, $\beta = c_1(\mathcal{F}_2(M))$ and $\gamma = c_1(\mathcal{F}_3(M))$. The coefficient on h is $\alpha + 2\beta + 3\gamma + 4$, which in any case must be congruent to 0 (mod p). Hence $\alpha \equiv -2\beta - 3\gamma - 4 \pmod{p}$. Substituting this and calculating the coefficient on h^3 yields a factorisation

$$-(\beta + \gamma + 1)(\beta + \gamma + 2)(2\beta + 8\gamma + 9) \equiv 0 \pmod{p},$$

hence $\beta \equiv -\gamma - 1$, $-\gamma - 2$ or $-4\gamma - \frac{9}{2}$. Substituting $\beta \equiv -\gamma - 1$ into the expression for the coefficient on h^2 yields

$$-3\gamma^2 - 6\gamma - 5 \equiv 0 \pmod{p}$$

so that

$$\gamma \equiv -1 \pm \frac{1}{3}u \pmod{p},$$

where $u \in \mathbb{Z}$ is an integer such that $u^2 \equiv -6$. Substituting this and $\beta \equiv -\gamma - 1$ into the expression for the coefficient on h^4 gives us $\frac{7}{3}$, which must be congruent to 0 (mod p). But this contradicts our assumption that $p \geq 11$. Similarly, substituting $\beta \equiv -\gamma - 2$ into the expression for the coefficient on h^2 again yields

$$-3\gamma^2 - 6\gamma - 5 \equiv 0 \pmod{p},$$

and substituting these results and calculating the coefficient on h^4 again gives us $\frac{7}{3}$, a contradiction. Hence we are left with the case $\beta \equiv -4\gamma - \frac{9}{2}$. Substituting this and calculating the coefficient on h^2 yields

$$-30\gamma^2 - 60\gamma - \frac{125}{4} \equiv 0 \pmod{p}$$

so that

$$\gamma \equiv -1 \pm \frac{1}{12}u \pmod{p}.$$

Substituting this and $\beta \equiv -4\gamma - \frac{9}{2}$ into the expression for the coefficient on h^4 yields $-\frac{77}{192}$, which is never congruent to zero so long as $p \neq 7$ or 11. If we demand that $p \geq 11$ and $r \geq 6$ however, then the coefficient on h^5 must be congruent to 0 (mod p), and one readily calculates this to be $\pm \frac{35}{96}u$, which is impossible. This completes the proof. \square

3.9. Discussion

While the techniques of the last three sections do produce new bounds on p and r for which there exist kE -modules having certain small Jordan types, one may well ask whether or not these new bounds are sharp. We now give several reasons why this is likely not the case.

We first note that very few stable Jordan types having only two blocks are compatible with the conjectures given in Section 2.8. For example, Rickard's Conjecture 2.25 would rule out the realisation of all stable Jordan types $[a][b]$ with $a \neq b$ except $[2][1]$, $[p-1][p-2]$ and $[p-1][1]$. The first two are actually realised in rank two by the modules kE/J^2 and J^2 , respectively, although Conjecture 2.28 would rule out their realisation in higher ranks. The stable Jordan type $[p-1][1]$ is realised in any rank by the module $k \oplus \Omega(k)$ per the discussion following Theorem 2.18. Both Rickard and Suslin's conjectures would rule out the realisation of all stable Jordan types $[a]^2$ except $[1]^2$ and $[p-1]^2$. The latter are realised in any rank by the modules $k \oplus k$ and $\Omega(k) \oplus \Omega(k)$, respectively.

A more conceptual argument against the sharpness of our bounds is the observation that the theory of Chern classes for vector bundles applies equally well to arbitrary nilvarieties of constant Jordan type. Specifically, a *nilvariety of constant Jordan type* is a subspace of $\text{Mat}_n(k)$ spanned by a set of nilpotent matrices A_1, \dots, A_r such that every non-zero linear combination of the A_i has the same Jordan canonical form. The point we wish to emphasise here is that the matrices A_i are not required to commute with one another. In other words, because the vector bundle techniques of Benson and Pevtsova do not exploit the commutative structure of kE , it is unlikely that any approach based solely upon them will yield sharp information about kE -modules of constant Jordan type.

Finally, perhaps the most convincing evidence against the sharpness of our bounds is Benson's Theorem 6.1, which was proved after our results had already been obtained. One of its consequences is the following, which appeared as Corollary 1.4 of [6]. In particular, for $r \geq 2$, it rules out the realisation of any stable Jordan type $[a][b]$ with $2 \leq a, b \leq p-2$.

Corollary 3.24. *If $r \geq 2$ and a kE -module has constant Jordan type containing no blocks of length 1 or $p - 1$, then the number of Jordan blocks of length less than p is divisible by p .*

PROOF. Recall from the proof of Theorem 2.18 that we have

$$\Omega(M) \downarrow_{\langle g_\alpha \rangle} \oplus P_{M \downarrow_{\langle g_\alpha \rangle}} \cong \Omega(M \downarrow_{\langle g_\alpha \rangle}) \oplus P_{M \downarrow_{\langle g_\alpha \rangle}}$$

for all non-zero $\alpha \in \mathbb{A}^r(k)$. The number of Jordan blocks of length p in $P_{M \downarrow_{\langle g_\alpha \rangle}}$ is equal to the total number of Jordan blocks of M by the way one computes projective covers of $k(\mathbb{Z}/p)$ -modules. Moreover, $\Omega(M \downarrow_{\langle g_\alpha \rangle})$ is projective free, and the number of Jordan blocks of length p in $P_{M \downarrow_{\langle g_\alpha \rangle}}$ is divisible by p since $r \geq 2$. This shows that the number of Jordan blocks of length p in $\Omega(M)$ is congruent modulo p to minus the total number of Jordan blocks of M .

Now observe by the remarks following Theorem 2.18 that if M contains no Jordan blocks of length $p - 1$, then $\Omega^{-1}(M)$ contains no Jordan blocks of length one. Replacing M by $\Omega^{-1}(M)$ in the above argument and applying Theorem 6.1, it follows that the total number of Jordan blocks of length p in M is divisible by p . By the same theorem, the total number of Jordan blocks of M is divisible by p , hence the number of remaining Jordan blocks must also be divisible by p . \square

In light of the preceding remarks, it is likely better to view the results of this chapter in the context of arbitrary nilvarieties of constant Jordan type, a class of objects for which it remains unknown whether or not the bounds we have provided are sharp.

CHAPTER 4

Constant image layers and vector bundles for W -modules

This chapter expands on the work of Carlson, Friedlander and Suslin [17] related to modules with the constant image property. Our goal is to study the unique submodule of a kE -module that is maximal with respect to having the constant image property. This submodule is known to be well behaved when E has rank two, as it coincides with the module's generic kernel. (See Chapter 5.) The best we are able to do here is show that every kE -module has a natural filtration with respect to these submodules in which almost all of the filtered subquotients are semisimple.

We also prove an interesting lemma relating the functors \mathcal{F}_i (see Section 3.3) to powers of the radical of a module having the constant image property. After introducing the class of W -modules in rank two, which were also defined in [17], we use this lemma to calculate the images of all W -modules under the \mathcal{F}_i .

4.1. The constant image property

The constant image property was introduced by Carlson, Friedlander and Pevtsova [16] and was investigated further by Carlson, Friedlander and Suslin [17]. We maintain our notation as in Chapter 2.

Definition 4.1. A kE -module M has the *constant image property* if $\text{lm}(X_\alpha, M)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$.

Note that M has the constant image property if and only if $X_\alpha M = \text{Rad}(M)$ for all non-zero $\alpha \in \mathbb{A}^r(k)$. To see this, suppose that

$$m = X_1 m_1 + \cdots + X_r m_r \in \text{Rad}(M)$$

with $m_1, \dots, m_r \in M$. If M has the constant image property, then for each $2 \leq i \leq r$ there exists $m'_i \in M$ such that $X_1 m'_i = X_i m_i$, hence $m = X_1(m_1 + \sum_{i=2}^r m'_i)$. It follows by another application of the constant image property that $m \in \text{lm}(X_\alpha, M)$ for all non-zero $\alpha \in \mathbb{A}^r(k)$.

As was the case with the constant Jordan type property (see Definition 2.14), it is not clear from the definition that the constant image property is independent of the

choice of generators g_1, \dots, g_r of E . The following proposition, which appeared as Proposition 2.6 (3) of [17], shows that this is indeed the case.

Proposition 4.2. *A kE -module M has the constant image property if and only if $\text{Im}(x, M)$ is independent of $x \in J \setminus J^2$.*

PROOF. One direction is clear. So suppose $x \in J \setminus J^2$ and write $x = X_\alpha + y$ with $\alpha \in \mathbb{A}^r(k)$ non-zero and $y \in J^2$. We have

$$\text{Rad}(M) = X_\alpha M \subseteq xM + yM \subseteq \text{Rad}(M)$$

by the constant image property, hence $xM + yM = \text{Rad}(M)$. Since $yM \subseteq \text{Rad}^2(M)$, it follows from Nakayama's lemma that $xM = \text{Rad}(M)$. \square

A nice feature of the constant image property is that it also remains invariant under field extensions. The following appeared as Proposition 2.5 (2) of [17].

Proposition 4.3. *A kE -module M has the constant image property if and only if for every field extension K/k , $\text{Im}(X_\alpha, M \otimes_k K)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(K)$.*

Our next goal is to show that every kE -module with the constant image property also has constant Jordan type, a fact recorded as Proposition 2.8 of [17].

Proposition 4.4. *If M is a kE -module with the constant image property and α is a non-zero point in $\mathbb{A}^r(k)$, then $X_\alpha^i M = \text{Rad}^i(M)$ for all $i \geq 0$.*

PROOF. We proceed by induction on i , the case $i = 1$ being clear by the constant image property. If $i \geq 1$ and $X_\alpha^i M = \text{Rad}^i(M)$, then by the commutativity of kE and the constant image property we have

$$X_\alpha^{i+1} M = X_\alpha \cdot X_\alpha^i M = X_\alpha \cdot J^i M = J^i \cdot X_\alpha M = J^i \cdot JM = \text{Rad}^{i+1}(M)$$

as required. \square

Proposition 4.5. *If M is a kE -module with the constant image property, then M has constant Jordan type.*

PROOF. By Proposition 4.4, the rank of X_α^i acting on M is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$ for all $i \geq 0$. The proof now follows after noting that the Jordan type of a nilpotent matrix is completely determined by the ranks of its powers. \square

We end this section by recording some facts about the constant image property that follow directly from the definition and Proposition 4.4.

Proposition 4.6. *The class of kE -modules with the constant image property is closed under taking finite direct sums, quotients and radicals.*

4.2. Constant image layers

Proposition 4.7. *If M is a kE -module and N, N' are submodules of M having the constant image property, then $N + N'$ has the constant image property.*

PROOF. We have $N + N' \cong (N \oplus N')/\text{Ker}(\phi)$, where $\phi: N \oplus N' \rightarrow N + N'$ is the surjection $(n, n') \mapsto n + n'$. The proof now follows from Proposition 4.6. \square

Corollary 4.8. *Every (non-zero) kE -module contains a unique (non-zero) submodule that is maximal with respect to having the constant image property.*

PROOF. This follows from Proposition 4.7 and the fact that for any kE -module M , $\text{Soc}(M)$ has the constant image property. \square

Definition 4.9. In light of Corollary 4.8, we denote the maximal submodule of M with the constant image property by $\mathcal{I}(M)$.

Definition 4.10. Let M be a kE -module. We define the *constant image series* of M inductively by $\mathcal{I}^0(M) = 0$ and $\mathcal{I}^n(M)/\mathcal{I}^{n-1}(M) = \mathcal{I}(M/\mathcal{I}^{n-1}(M))$. For $n > 0$, we call the subquotients $\mathcal{I}^n(M)/\mathcal{I}^{n-1}(M)$ the *constant image layers* of M .

Our goal in this section is to show for every kE -module M that the constant image layers $\mathcal{I}^n(M)/\mathcal{I}^{n-1}(M)$ are semisimple for all $n \geq 2$. The key result is the following.

Lemma 4.11. *If M is a kE -module, then $\mathcal{I}(M/\mathcal{I}(M)) = \text{Soc}(M/\mathcal{I}(M))$.*

PROOF. Let N be the preimage of $\text{Soc}(M/\mathcal{I}(M))$ under the canonical surjection $M \rightarrow M/\mathcal{I}(M)$. If L is the preimage of $\mathcal{I}(M/\mathcal{I}(M))$ under this map, then we have $N \subseteq L \subseteq M$ since $\text{Soc}(M/\mathcal{I}(M))$ has the constant image property. This means that $\mathcal{I}(L) = \mathcal{I}(M)$ and $N/\mathcal{I}(M) = \text{Soc}(L/\mathcal{I}(M))$. Replacing M by L , it therefore suffices to show that if $M/\mathcal{I}(M)$ has the constant image property, then $M = N$. The statement is clear if M has the constant image property, so assume that $M \neq \mathcal{I}(M)$.

As a first step towards proving the statement, we claim there exists an element $m \in \mathcal{I}(M)$ and linearly independent $\alpha, \beta \in \mathbb{A}^r(k)$ such that $m \in \text{Im}(X_\alpha, N) \setminus \text{Im}(X_\beta, N)$. Suppose this is not the case. Then if $m \in \mathcal{I}(M)$, we have either $m \in \text{Im}(X_\alpha, N)$ for

all non-zero $\alpha \in \mathbb{A}^r(k)$ or $m \notin \text{Im}(X_\alpha, N)$ for any $\alpha \in \mathbb{A}^r(k)$. In the latter situation we cannot have $m \in \text{Rad}(N)$. To see this, suppose otherwise and write

$$m = X_1 n_1 + \cdots + X_r n_r \quad \text{for some } n_1, \dots, n_r \in N.$$

Then $m - (X_1 n_1 + \cdots + X_{r-1} n_{r-1}) \in \text{Im}(X_r, N)$, but $m \notin \text{Im}(X_r, N)$ by assumption, so we must have $X_1 n_1 + \cdots + X_{r-1} n_{r-1} \notin \text{Im}(X_r, N)$. Our original assumption then allows us to conclude that $X_1 n_1 + \cdots + X_{r-1} n_{r-1} \notin \text{Im}(X_\alpha, N)$ for any $\alpha \in \mathbb{A}^r(k)$. In particular, we have $X_1 n_1 + \cdots + X_{r-1} n_{r-1} \notin \text{Im}(X_{r-1}, N)$. Proceeding by induction, we eventually see that $X_1 n_1 \notin \text{Im}(X_2, N)$. Thus $X_1 n_1 \in \text{Rad}(N) \subseteq \mathcal{I}(M)$ and

$$X_1 n_1 \in \text{Im}(X_1, N) \setminus \text{Im}(X_2, N),$$

contradicting our original assumption. In summary, this shows that if $m \in \text{Rad}(N)$, then $m \in \text{Im}(X_\alpha, N)$ for all non-zero $\alpha \in \mathbb{A}^r(k)$. This means that N has the constant image property, and it follows that $N = \mathcal{I}(M)$. On the other hand, we assumed that M does not have the constant image property, so N properly contains $\mathcal{I}(M)$. This is a contradiction, and the claim now follows.

Having established the claim, we now suppose that $M \neq N$. Then because $M/\mathcal{I}(M)$ is not semisimple, its radical length d is greater than one. Since the constant image property is closed under taking radicals, one sees that $\text{Rad}^{d-2}(M/\mathcal{I}(M))$ also has the constant image property. Replacing M by $\text{Rad}^{d-2}(M) + \mathcal{I}(M)$ and N by

$$(N \cap \text{Rad}^{d-2}(M)) + \mathcal{I}(M),$$

we may therefore assume that $M/\mathcal{I}(M)$ has radical length two. Note that $M/\mathcal{I}(M)$ will continue to have a direct summand that is not semisimple. Because the constant image property is also closed under taking direct summands, we may assume further that $M/\mathcal{I}(M)$ contains no simple direct summand. These conditions imply that

$$\text{Rad}(M/\mathcal{I}(M)) = \text{Soc}(M/\mathcal{I}(M)) = N/\mathcal{I}(M).$$

Now let m, α, β be as described in the claim and choose $n \in N$ such that $m = X_\alpha n$. Then since $M/\mathcal{I}(M)$ has the constant image property and $n + \mathcal{I}(M)$ is in the radical of $M/\mathcal{I}(M)$, there exists $n' \in M$ such that $X_\beta(n' + \mathcal{I}(M)) = n + \mathcal{I}(M)$. We have

$$\begin{aligned} X_\beta X_\alpha n' &= X_\alpha X_\beta n' = X_\alpha(n + m_0) \quad \text{for some } m_0 \in \mathcal{I}(M) \\ &= m + X_\alpha m_0 = m + X_\beta m_1 \quad \text{for some } m_1 \in \mathcal{I}(M) \end{aligned}$$

since $\mathcal{I}(M)$ has the constant image property, thus $m = X_\beta(X_\alpha n' - m_1)$. But we also have $X_\alpha n' + \mathcal{I}(M) \in \text{Rad}(M/\mathcal{I}(M)) = N/\mathcal{I}(M)$, forcing $X_\alpha n' - m_1 \in N$. This contradicts our assumption that $m \notin \text{Im}(X_\beta, N)$, completing the proof. \square

Theorem 4.12. *For $n \geq 2$ we have $\mathcal{I}^n(M)/\mathcal{I}^{n-1}(M) = \text{Soc}(M/\mathcal{I}^{n-1}(M))$, hence all such constant image layers are semisimple.*

PROOF. For convenience, we write $\mathcal{I}^n = \mathcal{I}^n(M)$. By Lemma 4.11 we have

$$\begin{aligned} \mathcal{I}^n/\mathcal{I}^{n-1} &= \mathcal{I}(M/\mathcal{I}^{n-1}) \cong \mathcal{I}((M/\mathcal{I}^{n-2})/(\mathcal{I}^{n-1}/\mathcal{I}^{n-2})) \\ &= \mathcal{I}((M/\mathcal{I}^{n-2})/\mathcal{I}(M/\mathcal{I}^{n-2})) = \text{Soc}((M/\mathcal{I}^{n-2})/\mathcal{I}(M/\mathcal{I}^{n-2})) \quad \square \\ &= \text{Soc}((M/\mathcal{I}^{n-2})/(\mathcal{I}^{n-1}/\mathcal{I}^{n-2})) \cong \text{Soc}(M/\mathcal{I}^{n-1}). \end{aligned}$$

4.3. W -modules and the constant image property in rank two

As observed by Carlson, Friedlander and Suslin [17], the constant image property is particularly well behaved in the case where E has rank two. Specifically, the authors of [17] introduced the family of $k(\mathbb{Z}/p)^2$ -modules called W -modules. As we shall see in this section, every W -module has the constant image property, and what's more, a $k(\mathbb{Z}/p)$ -module has the constant image property if and only if it is the homomorphic image of some W -module.

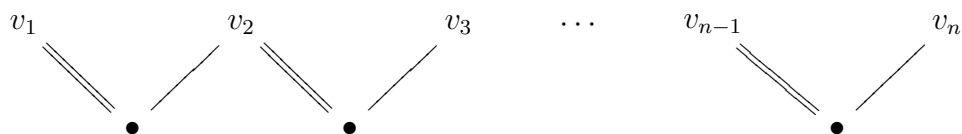
For the remainder of this thesis, unless otherwise specified, we shall assume that $r = 2$, that is, $E = \langle g_1, g_2 \rangle \cong (\mathbb{Z}/p)^2$, where g_1, g_2 are fixed commuting generators having order p . Recall that the Jacobson radical $J = J(kE)$ is generated by the elements $X_1 = g_1 - 1$ and $X_2 = g_2 - 1$.

Definition 4.13. Let n and d be positive integers such that $1 \leq d \leq n$ and $d \leq p$. If V is the free kE -module of rank n with generators v_1, \dots, v_n , we define $W_{n,d}$ to be the quotient V/U , where U is the kE -submodule of V generated by the elements

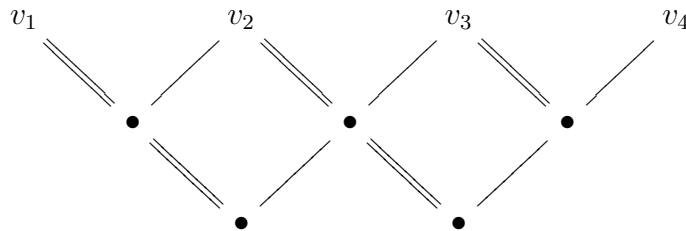
$$X_1 v_1, \quad X_2 v_n, \quad X_1^d v_i \text{ for } 1 \leq i \leq n, \quad X_2 v_i - X_1 v_{i+1} \text{ for } 1 \leq i \leq n-1.$$

Any kE -module of the form $W_{n,d}$ is called a W -module.

W -modules can be visualised using module diagrams as introduced in Section 2.4. For example, if p is any prime number and $n \geq 2$, then the module $W_{n,2}$ is represented by the diagram



Similarly, for $p \geq 3$, the module $W_{4,3}$ has diagram



Observe that both of the above diagrams have what can be roughly described as a ‘W’ shape, hence the terminology ‘W-module’.

It is easy to verify that every W -module has the constant image property, thus has constant Jordan type. The following appeared as Proposition 3.3 of [17].

Proposition 4.14. *If $1 \leq d \leq n$ and $d \leq p$, then the kE -module $W_{n,d}$ has the constant image property.*

PROOF. It suffices to show that $X_\alpha W_{n,d} = \text{Rad}(W_{n,d})$ for all non-zero $\alpha \in \mathbb{A}^2(k)$. Without loss of generality, we may assume that $\alpha = (1, \lambda)$ for some $\lambda \in k$. Observe that $\text{Rad}(W_{n,d})$ is generated by the elements $X_1 v_2, \dots, X_1 v_n$. We have

$$X_1 v_i = X_\alpha v_i - \lambda X_2 v_i = X_\alpha v_i - \lambda X_1 v_{i+1}.$$

Continuing in this way we obtain $X_1 v_i = \sum_{j=i}^n (-\lambda)^{j-i} X_\alpha v_j$ so that

$$X_\alpha W_{n,d} = X_1 W_{n,d} = \text{Rad}(W_{n,d}). \quad \square$$

Corollary 4.15. *If $1 \leq d \leq n$ and $d \leq p$, then $W_{n,d}$ has constant Jordan type $[d]^{n-d+1}[d-1] \dots [1]$.*

PROOF. The fact that $W_{n,d}$ has constant Jordan type follows from Proposition 4.5. Calculating the Jordan type is accomplished by calculating the Jordan type of X_1 on $W_{n,d}$, using the corresponding module diagram as illustrated above. \square

Not only do W -modules have the constant image property, but it can also be shown that every kE -module with the constant image property is realised as the homomorphic image of some $W_{n,d}$. In particular, we record the following theorem, which appeared as Theorem 5.4 of [17].

Theorem 4.16. *If M is a kE -module with the constant image property having radical length d , then there exists an integer $n \geq d$ and a surjective kE -module homomorphism $W_{n,d} \rightarrow M$.*

4.4. Vector bundles for W -modules

Motivated by the central role W -modules play in the theory of kE -modules with the constant image property, we now calculate the vector bundles $\mathcal{F}_i(W_{n,d})$ for $1 \leq i \leq d$. In [17, Proposition 6.4], the authors calculated the kernel bundle $\text{Ker } \theta_{W_{n,d}}$. Referring to the proof of Proposition 3.7, our calculation is in some sense a refinement of their result. We begin with a general lemma, which may be thought of as a vector bundle analog of Proposition 4.4.

Lemma 4.17. *If M is a kE -module with the constant image property in any rank r , then for all $0 \leq j < i \leq p$ we have $\mathcal{F}_i(M) \cong \mathcal{F}_{i-j}(\text{Rad}^j(M))$.*

PROOF. By the definition of the operator θ_M , the map $\theta_M^j: \widetilde{M}(-i) \rightarrow \widetilde{M}(-i+j)$ induces a natural map of vector bundles

$$\widetilde{M}(-i) \xrightarrow{\widehat{\theta}_M^j} \widetilde{\text{Rad}^j(M)}(-i+j).$$

We claim that this map is surjective. Note by Proposition 3.5 that its fibre at a point $x \in \mathbb{P}^{r-1}$ is

$$M \otimes_k k(x) \xrightarrow{X_{\bar{\alpha}}^j} \text{Rad}^j(M) \otimes_k k(x),$$

where $\bar{\alpha} \in \mathbb{P}^{r-1}(k(x))$ corresponds to the generic point of x . Since M has the constant image property, so does $M \otimes_k k(x)$ by Proposition 4.3, hence the map $X_{\bar{\alpha}}^j$ is surjective by Proposition 4.4. It now follows from Proposition 3.4 that $\widehat{\theta}_M^j$ is surjective.

Having established the claim, consider the composition of maps

$$\widetilde{M}(-i) \xrightarrow{\widehat{\theta}_M^j} \widetilde{\text{Rad}^j(M)}(-i+j) \xrightarrow{\theta_{\text{Rad}^j(M)}^{i-j}} \widetilde{\text{Rad}^j(M)}.$$

Since the map on the left is surjective, the image of the composition is equal to that of the map on the right. But the image of the composition is also isomorphic to $\text{Im}\{\theta_M^j: \widetilde{M}(-i) \rightarrow \widetilde{M}\}$. It follows that

$$\begin{aligned} \mathcal{F}_i(M) &= \frac{\text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}}{\text{Ker } \theta_M \cap \text{Im } \theta_M^i} \cong \frac{\text{Ker } \theta_M \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j-1}}{\text{Ker } \theta_M \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j}} \cong \frac{\text{Ker } \theta_{\text{Rad}^j(M)} \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j-1}}{\text{Ker } \theta_{\text{Rad}^j(M)} \cap \text{Im } \theta_{\text{Rad}^j(M)}^{i-j}} \\ &= \mathcal{F}_{i-j}(\text{Rad}^j(M)) \end{aligned}$$

as required. \square

Returning to the rank two case, we recall that Grothendieck [28] has classified the vector bundles on \mathbb{P}^1 . In particular, every such bundle is a direct sum of line bundles

$$\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_t),$$

where the integers n_1, \dots, n_t are uniquely determined up to reordering. Given this classification, we now present the main theorem of this section.

Theorem 4.18. *If $1 \leq d \leq n$ and $d \leq p$, then $\mathcal{F}_i(W_{n,d}) \cong \mathcal{O}(-n+i)$ for $1 \leq i \leq d-1$, and $\mathcal{F}_d(W_{n,d}) \cong \mathcal{O}^{\oplus(n-d+1)}$.*

PROOF. We proceed by induction on d , the case $d = 1$ being trivial. So suppose $d > 1$. Since the trivial bundle $\widetilde{W}_{n,d}$ has a filtration with filtered quotients $\mathcal{F}_i(M)(j)$ for $0 \leq j < i \leq d$ by Proposition 3.7, we have

$$1 = c(\widetilde{W}_{n,d}) = \prod_{0 \leq j < i \leq d} \mathcal{F}_i(W_{n,d})(j).$$

Comparing the first Chern numbers using Lemma 3.10 gives us

$$0 = \sum_{i=1}^{d-1} (i \cdot c_1(\mathcal{F}_i(W_{n,d})) + \frac{1}{2}i(i-1)) + \sum_{j=0}^{d-1} j \cdot c_1(\mathcal{F}_d(W_{n,d})). \quad (4.1)$$

Note that $\text{Rad}(W_{n,d})$ is also a W -module, being isomorphic to $W_{n-1,d-1}$. Hence by induction and Lemma 4.17 we have

$$\mathcal{F}_i(W_{n,d}) \cong \mathcal{F}_{i-1}(\text{Rad}(W_{n,d})) \cong \mathcal{F}_{i-1}(W_{n-1,d-1}) \cong \mathcal{O}(-n+i)$$

for $2 \leq i \leq d-1$ and

$$\mathcal{F}_d(W_{n,d}) \cong \mathcal{F}_{d-1}(\text{Rad}(W_{n,d})) \cong \mathcal{F}_{d-1}(W_{n-1,d-1}) \cong \mathcal{O}^{\oplus(n-d+1)}.$$

Substituting this into (4.1) and simplifying yields

$$0 = c_1(\mathcal{F}_1(W_{n,d})) + n - 1$$

so that $\mathcal{F}_1(W_{n,d}) \cong \mathcal{O}(-n+1)$. This completes the proof. \square

The following corollary shows that the functor $\mathcal{F}_1: \mathbf{cJt}(kE) \rightarrow \mathbf{Vec}(\mathbb{P}^1)$ is essentially surjective.

Corollary 4.19. *If \mathcal{F} is a vector bundle on \mathbb{P}^1 , then there exists a finitely generated kE -module M of constant Jordan type such that $\mathcal{F}_1(M) \cong \mathcal{F}$.*

PROOF. By the above remarks, it suffices to realise all line bundles on \mathbb{P}^1 since \mathcal{F}_1 distributes over direct sums. By Theorem 4.18, we have $\mathcal{F}_1(W_{n,2}) \cong \mathcal{O}(-n+1)$ for all $n \geq 2$ and $\mathcal{F}_1(k) \cong \mathcal{F}_1(W_{1,1}) \cong \mathcal{O}$. To realise the line bundles with positive first Chern numbers, we use Proposition 3.9 to obtain

$$\mathcal{F}_1(W_{n,2}^*) \cong \mathcal{F}_1(W_{n,2})^\vee \cong \mathcal{O}(-n+1)^\vee \cong \mathcal{O}(n-1) \quad \text{for all } n \geq 2. \quad \square$$

Generic kernels and a question of Carlson, Friedlander and Suslin

This chapter provides a thorough treatment of Question 8.11 of Carlson, Friedlander and Suslin [17] regarding generic kernels for modules of constant Jordan type in rank two. Aside from giving a detailed answer to this question, we derive a particularly nice duality formula for a module of constant rank that relates its generic kernel to that of its dual. We make a special point of showing that this formula is independent of the Hopf algebra structure on kE . The exposition of background material closely follows that of [17], where all relevant concepts were originally introduced.

5.1. Generic kernels

Throughout this chapter we continue to require that E has rank two, unless otherwise specified. We now introduce the primary structure of interest throughout the rest of the thesis, namely the generic kernel of a kE -module, which was defined by Carlson, Friedlander and Suslin [17]. There, the authors showed that the generic kernel $\mathfrak{K}(M)$ of a kE -module M coincides with the maximal submodule of M having the constant image property. (See Section 4.2.)

Let M be a finite dimensional kE -module. Note for any non-zero point

$$\alpha = (\lambda_1, \lambda_2) \in \mathbb{A}^2(k)$$

that $\text{Ker}(X_\alpha, M)$ is uniquely determined by the class $\bar{\alpha} = [\lambda_1 : \lambda_2] \in \mathbb{P}^1(k)$. If S is a subset of $\mathbb{P}^1(k)$, we denote by ${}_S M$ the vector space $\sum_{\bar{\alpha} \in S} \text{Ker}(X_\alpha, M)$. Observe that ${}_S M$ is in fact a kE -submodule of M , because kE is a commutative ring. Recall that a subset $S \subseteq \mathbb{P}^1(k)$ is *cofinite* if its complement in $\mathbb{P}^1(k)$ is a finite set.

Definition 5.1 (Carlson, Friedlander and Suslin [17]). If M is a finite dimensional kE -module, we define the *generic kernel* of M to be the submodule

$$\mathfrak{K}(M) = \bigcap_{S \subseteq \mathbb{P}^1(k) \text{ cofinite}} {}_S M$$

of M .

Note that there always exists a cofinite $S \subseteq \mathbb{P}^1(k)$ such that $\mathfrak{K}(M) = {}_S M$. To see this, suppose otherwise and observe from the definition that $\mathfrak{K}(M) \subseteq {}_{\mathbb{P}^1(k)} M$. Since $\mathfrak{K}(M) \neq {}_{\mathbb{P}^1(k)} M$, there exists a cofinite $S_1 \subseteq \mathbb{P}^1(k)$ such that ${}_{\mathbb{P}^1(k)} M \not\subseteq_{S_1} M$. Similarly, because $\mathfrak{K}(M) \neq {}_{S_1} M$, there exists a cofinite $S_2 \subseteq \mathbb{P}^1(k)$ such that ${}_{S_1} M \not\subseteq_{S_2} M$. Writing $S_1 = \mathbb{P}^1(k) \setminus T_1$ and $S_2 = \mathbb{P}^1(k) \setminus T_2$ with $T_1, T_2 \subseteq \mathbb{P}^1(k)$ finite, we then have

$${}_{\mathbb{P}^1(k)} M \supsetneq {}_{\mathbb{P}^1(k) \setminus T_1} M \supsetneq {}_{\mathbb{P}^1(k) \setminus (T_1 \cup T_2)} M \supsetneq \mathfrak{K}(M).$$

Continuing in this way, we obtain a strictly descending chain of submodules

$${}_{\mathbb{P}^1(k)} M \supsetneq {}_{\mathbb{P}^1(k) \setminus T_1} M \supsetneq {}_{\mathbb{P}^1(k) \setminus (T_1 \cup T_2)} M \supsetneq \cdots \supsetneq {}_{\mathbb{P}^1(k) \setminus (T_1 \cup \cdots \cup T_n)} M \supsetneq \cdots,$$

contradicting the fact that M is finite dimensional.

Using the identity ${}_T({}_S M) = {}_{S \cap T} M$ for all cofinite $S, T \subseteq \mathbb{P}^1(k)$, one also sees from the definition that $\mathfrak{K}(\mathfrak{K}(M)) = \mathfrak{K}(M)$.

The following theorem appeared as Theorem 7.10 of [17]. It provides an essential link between the generic kernel and the constant image property.

Theorem 5.2. *If M is a kE -module, then the generic kernel $\mathfrak{K}(M)$ has the constant image property. Moreover, if N is any submodule of M having the constant image property, then $N \subseteq \mathfrak{K}(M)$.*

PROOF. We shall prove the first statement to demonstrate the geometric nature of the argument. The proof of the second statement is more difficult, and we leave the reader to study its proof in [17].

It suffices to show that $\text{Im}(X_\alpha, \mathfrak{K}(M)) = \text{Im}(X_1, \mathfrak{K}(M))$ for each non-zero $\alpha \in \mathbb{A}^2(k)$. Let $S \subseteq \mathbb{P}^1(k)$ be cofinite such that $\mathfrak{K}(M) = {}_S M = \sum_{\bar{\beta} \in S} \text{Ker}(X_\beta, M)$. We may assume that S contains neither $\bar{\alpha}$ nor $[0 : 1]$, the latter corresponding to the element $X_1 \in J$. Let m be a generator of $\mathfrak{K}(M)$ lying in $\text{Ker}(X_\beta, M)$ for some $\bar{\beta} \in S$, and let $\beta \in \mathbb{A}^2(k)$ be any point lying above $\bar{\beta}$. Since α and β are linearly independent, X_α and X_β generate J . It follows that

$$\begin{aligned} \text{Rad}(kE.m) &= J.m = (X_\alpha, X_\beta).m = X_\alpha.kE.m + X_\beta.kE.m \\ &= X_\alpha.kE.m + kE.X_\beta.m = X_\alpha.kE.m. \end{aligned}$$

By a similar argument we have $\text{Rad}(kE.m) = X_1.kE.m$. Summing over all generators of the above form completes the proof. \square

5.2. Generic kernels for modules of constant rank

For the moment, we temporarily relax our condition that E has rank two in order to introduce modules of constant rank in a more general context.

Definition 5.3. A kE -module M has *constant rank* if $\text{rank}(X_\alpha, M)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$. In particular, if M has constant Jordan type then M has constant rank.

Much of our future discussion will involve showing that certain duality results are independent of the Hopf algebra structure on kE . With this goal in mind we record the following lemma, which may be found in Jantzen [31, I.2.4].

Lemma 5.4. *Let $\tau: kE \rightarrow kE$ be the antipode for any Hopf algebra structure on kE . If $x \in J$ and α is the point in $\mathbb{A}^r(k)$ such that $x \equiv X_\alpha \pmod{J^2}$, then*

$$\tau(x) \equiv -X_\alpha \pmod{J^2}.$$

We now give a proposition that implies, in particular, that M has constant rank if and only if M^* has constant rank. The proof is similar to that of Proposition 2.11. The main feature here is that we avoid fixing the Hopf algebra structure on kE .

Proposition 5.5. *Let $\tau: kE \rightarrow kE$ be the antipode for any Hopf algebra structure on kE . Let M be a kE -module and let M_τ^* denote the dual kE -module structure on $\text{Hom}_k(M, k)$ induced by τ . If α is a point in $\mathbb{A}^r(k)$ such that the rank of X_α on M is maximal amongst all points in $\mathbb{A}^r(k)$, then the rank of X_α on M_τ^* is equal to the rank of X_α on M .*

PROOF. Observe that $X_\alpha + \tau(X_\alpha) \in J^2$ for all $\alpha \in \mathbb{A}^r(k)$ by Lemma 5.4. If X_α achieves the maximal rank on M , then it follows from the second part of Theorem 2.5 that $\text{rank}(X_\alpha, M) = \text{rank}(\tau(X_\alpha), M)$. The proof is complete after noting that the representation affording M_τ^* is obtained from the representation affording M by precomposing with τ and then composing with the transpose map. \square

Returning to the case in which E has rank two, our next goal is to show that $\mathfrak{K}(M)$ is especially well behaved whenever M has constant rank. The following appeared as Proposition 7.6 of [17].

Lemma 5.6. *Let M be a finite dimensional kE -module.*

- (1) *For any $\bar{\alpha} \in \mathbb{P}^1(k)$, we have $\text{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$ if and only if the rank of X_α is maximal amongst all points in $\mathbb{A}^2(k)$.*

- (2) *The set of points $\bar{\alpha} \in \mathbb{P}^1(k)$ such that X_α achieves the maximal rank on M is the largest subset $S \subseteq \mathbb{P}^1(k)$ such that $\mathfrak{K}(M) = {}_S M$.*

PROOF. Write $\mathfrak{K}(M) = {}_S M$ for some cofinite $S \subseteq \mathbb{P}^1(k)$. Since S is an infinite set, it is dense in $\mathbb{P}^1(k)$, hence intersects every open set of $\mathbb{P}^1(k)$ non-trivially. Now the set of points $\bar{\alpha} \in \mathbb{P}^1(k)$ such that $\text{rank}(X_\alpha, M)$ is maximal forms an open subset of $\mathbb{P}^1(k)$. Thus there exists a point $\bar{\alpha} \in \mathbb{P}^1(k)$ such that $\text{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$ and such that X_α has maximal rank on M . For any $\bar{\beta} \in \mathbb{P}^1(k)$ we then have

$$\begin{aligned} \dim_k \text{Ker}(X_\beta, M) &\geq \dim_k \text{Ker}(X_\alpha, M) \\ &= \dim_k \text{Ker}(X_\alpha, \mathfrak{K}(M)) \\ &= \dim_k \text{Ker}(X_\beta, \mathfrak{K}(M)). \end{aligned}$$

The inequality follows from the fact that $\text{rank}(X_\alpha, M)$ is maximal, the first equality follows from the fact that $\text{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$, and the second equality follows from the fact that $\mathfrak{K}(M)$ has the constant image property. Note that X_β has maximal rank on M if and only if the inequality is an equality, which is true if and only if $\text{Ker}(X_\beta, M) \subseteq \mathfrak{K}(M)$. This proves part (i). Part (ii) follows by the same argument. \square

As an immediate consequence, we obtain the following characterisation of the constant rank property in terms of the generic kernel.

Theorem 5.7. *A kE -module M has constant rank if and only if $\mathfrak{K}(M) = {}_{\mathbb{P}^1(k)} M$. In this case we have $\text{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$ for all non-zero $\alpha \in \mathbb{A}^2(k)$.*

5.3. The generic kernel filtration and the main question

The goal of this section is to introduce the main question concerning generic kernels and modules of constant Jordan type that we wish to address in this chapter.

Let M be a kE -module and N a submodule of M . If I is an ideal of kE and $i \in \mathbb{N}$, then we denote by $I^{-i}N$ the set of elements $m \in M$ satisfying $I^i m \subseteq N$. Similarly, for $x \in kE$, we denote by $x^{-i}N$ the set of elements $m \in M$ for which $x^i m \in N$. It is immediately verified that both of these constructions are submodules of M , because kE is a commutative ring.

Armed with Theorem 5.7, we now introduce the main proposition of the section. It originally appeared as part of [17, Theorem 8.10]. We provide an alternative proof here.

Proposition 5.8. *If M is a kE -module of constant rank and α is a non-zero point in $\mathbb{A}^2(k)$, then $X_\alpha^{-i}\mathfrak{K}(M) = J^{-i}\mathfrak{K}(M)$ for all $i \geq 0$.*

PROOF. The leftwards containment is clear for each $i \geq 0$, for if $m \in M$ satisfies $J^i m \subseteq \mathfrak{K}(M)$, then we obviously have $X_\alpha^i m \in \mathfrak{K}(M)$. We are therefore left to prove the reverse containment, proceeding by induction on i . For $i = 1$, let $m \in M$ such that $X_\alpha m \in \mathfrak{K}(M)$. Given $\beta \in \mathbb{A}^2(k)$, we then have $X_\beta X_\alpha m \in J\mathfrak{K}(M)$. By Theorem 5.2 and the constant image property, there exists $m' \in \mathfrak{K}(M)$ such that

$$X_\alpha m' = X_\beta X_\alpha m = X_\alpha X_\beta m.$$

It follows by Theorem 5.7 that $m' - X_\beta m \in \mathbf{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$, hence $X_\beta m$ is also in $\mathfrak{K}(M)$. Since the elements X_β generate J , this shows that $Jm \subseteq \mathfrak{K}(M)$.

Now let $i > 1$ and assume by induction that $X_\alpha^{-i}\mathfrak{K}(M) = J^{-i}\mathfrak{K}(M)$. Letting $m \in M$ satisfy $X_\alpha^i m \in \mathfrak{K}(M)$, we then have $X_\alpha^i X_\alpha m \in \mathfrak{K}(M)$, hence

$$X_\alpha m \in X_\alpha^{-i}\mathfrak{K}(M) = J^{-i}\mathfrak{K}(M).$$

It follows that $X_\alpha J^i m \subseteq \mathfrak{K}(M)$ so that $J^i m \in X_\alpha^{-1}\mathfrak{K}(M) = J^{-1}\mathfrak{K}(M)$ by the opening step of the induction. This shows that $J^{i+1}m = JJ^i m \subseteq \mathfrak{K}(M)$ as required. \square

The following corollary will be used implicitly in Section 5.8 whenever we invoke the duality provided by Theorem 5.23.

Corollary 5.9. *If M is a kE -module of constant rank, then $M = J^{-p+1}\mathfrak{K}(M)$ and $J^p\mathfrak{K}(M) = 0$.*

PROOF. For any non-zero $\alpha \in \mathbb{A}^2(k)$ we have $X_\alpha^p M = 0$, hence

$$X_\alpha^{p-1}M \subseteq \mathbf{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$$

by Theorem 5.7. It follows that $M \subseteq X_\alpha^{-p+1}\mathfrak{K}(M) = J^{-p+1}\mathfrak{K}(M)$ by Proposition 5.8. Finally, we have $J^p\mathfrak{K}(M) = 0$ by Proposition 4.4. \square

If M is a kE -module of constant rank, then Corollary 5.9, Theorem 5.2 and Proposition 4.6 show that M has a filtration

$$0 = J^p\mathfrak{K}(M) \subseteq J^{p-1}\mathfrak{K}(M) \subseteq \cdots \subseteq J^{-p+2}\mathfrak{K}(M) \subseteq J^{-p+1}\mathfrak{K}(M) = M \quad (5.1)$$

in which $J^i\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$. Moreover, if M itself has constant Jordan type, then the previous statement is also true for $i = -p + 1$. We call (5.1) the *generic kernel filtration* of M . The above observations lead naturally to our main question, which appeared as Question 8.11 of [17].

Question 5.10. If M is a finite dimensional kE -module of constant Jordan type, is it true that $J^{-i}\mathfrak{K}(M)$ also has constant Jordan type for all $i \geq 0$?

As we shall see later in the chapter, Question 5.10 has an affirmative answer whenever $p = 3$ or $J^2\mathfrak{K}(M) = 0$. We shall also see that these restrictions are sharp, that is, we will construct a counterexample to Question 5.10 in the first interesting case, namely that in which $p \geq 5$, $J^2\mathfrak{K}(M) \neq 0$ and $J^3\mathfrak{K}(M) = 0$.

5.4. Generic images

In order to prove statements about duality and generic kernels in the sequel, we now introduce the concept of the generic image of a kE -module. As with the preceding material in this chapter, the generic image was defined and investigated by Carlson, Friedlander and Suslin [17].

Let M be a finite dimensional kE -module. Observe for any non-zero point $\alpha \in \mathbb{A}^2(k)$ that $\mathfrak{lm}(X_\alpha, M)$ is uniquely determined by the class $\bar{\alpha} \in \mathbb{P}^1(k)$. If S is a subset of $\mathbb{P}^1(k)$, we denote by M_S the vector space $\bigcap_{\bar{\alpha} \in S} \mathfrak{lm}(X_\alpha, M)$. As before, the fact that kE is a commutative ring implies that M_S is a kE -submodule of M .

Definition 5.11 (Carlson, Friedlander and Suslin [17]). If M is a finite dimensional kE -module, then the *generic image* of M is defined to be the submodule

$$\mathfrak{J}(M) = \sum_{S \subseteq \mathbb{P}^1(k) \text{ cofinite}} M_S$$

of M .

As was the case with generic kernels, there always exists a cofinite $S \subseteq \mathbb{P}^1(k)$ such that $\mathfrak{J}(M) = M_S$. To see this, suppose otherwise and observe from the definition that $M_{\mathbb{P}^1(k)} \subseteq \mathfrak{J}(M)$. Since $\mathfrak{J}(M) \neq M_{\mathbb{P}^1(k)}$, there exists a cofinite $S_1 \subseteq \mathbb{P}^1(k)$ such that $M_{S_1} \subsetneq M_{\mathbb{P}^1(k)}$. Similarly, because $\mathfrak{J}(M) \neq M_{S_1}$, there exists a cofinite $S_2 \subseteq \mathbb{P}^1(k)$ such that $M_{S_2} \subsetneq M_{S_1}$. Writing $S_1 = \mathbb{P}^1(k) \setminus T_1$ and $S_2 = \mathbb{P}^1(k) \setminus T_2$ with $T_1, T_2 \subseteq \mathbb{P}^1(k)$ finite, we then have

$$M_{\mathbb{P}^1(k)} \subsetneq M_{\mathbb{P}^1(k) \setminus T_1} \subsetneq M_{\mathbb{P}^1(k) \setminus (T_1 \cup T_2)} \subsetneq \mathfrak{J}(M).$$

Continuing in this way, we obtain a strictly ascending chain of submodules

$$M_{\mathbb{P}^1(k)} \subsetneq M_{\mathbb{P}^1(k) \setminus T_1} \subsetneq M_{\mathbb{P}^1(k) \setminus (T_1 \cup T_2)} \subsetneq \cdots \subsetneq M_{\mathbb{P}^1(k) \setminus (T_1 \cup \cdots \cup T_n)} \subsetneq \cdots,$$

contradicting the fact that M is finite dimensional.

Given any Hopf algebra structure on kE , recall that if M is a kE -module and N is a submodule of M , then the orthogonal complement

$$N^\perp = \{f \in \text{Hom}_k(M, k) \mid f(n) = 0 \text{ for all } n \in N\}$$

is naturally a kE -submodule of $\text{Hom}_k(M, k)$, and we have $(N^\perp)^\perp \cong N$. The following appeared as Proposition 8.4 of [17]. It relates the generic image to the generic kernel through duality and orthogonality.

Proposition 5.12. *If M is a kE -module and $M^{\tilde{*}}$ is the Lie theoretic dual of M (see Section 2.2), then*

$$\mathfrak{J}(M) \cong \mathfrak{K}(M^{\tilde{*}})^\perp \quad \text{and} \quad \mathfrak{K}(M) \cong \mathfrak{J}(M^{\tilde{*}})^\perp.$$

PROOF. If $\phi: M \rightarrow M$ is any k -linear map, we denote by ϕ^* the dual linear map $\text{Hom}_k(M, k) \rightarrow \text{Hom}_k(M, k)$ given by $\phi^*(f) = f \circ \phi$. In other words, ϕ^* is the image of ϕ under the transpose operation. We have

$$\text{Im}(\phi)^\perp = \{f: M \rightarrow k \mid f \circ \phi = 0\} = \text{Ker}(\phi^*).$$

In the case where ϕ is given by the action of X_α on M , we have $X_\alpha^* = -\tilde{\sigma}(X_\alpha)$, where $\tilde{\sigma}$ is the Lie theoretic antipode of kE . It follows that

$$\text{Im}(X_\alpha, M)^\perp = \text{Ker}(X_\alpha, M^{\tilde{*}}).$$

A standard result in linear algebra states that if $\{V_i\}$ is any family of subspaces of a finite dimensional vector space V , then

$$\left(\bigcap_i V_i\right)^\perp = \sum_i V_i^\perp$$

as subspaces of V^* . For any cofinite $S \subseteq \mathbb{P}^1(k)$, we thus have

$$\left(\bigcap_{\bar{\alpha} \in S} \text{Im}(X_\alpha, M)\right)^\perp = \sum_{\bar{\alpha} \in S} \text{Ker}(X_\alpha, M^{\tilde{*}}).$$

Choosing S such that $\mathfrak{K}(M) = {}_S M$ and $\mathfrak{J}(M) = M_S$, we then have $\mathfrak{J}(M)^\perp \cong \mathfrak{K}(M^{\tilde{*}})$, establishing the first isomorphism. The second follows by replacing M with $M^{\tilde{*}}$. \square

Remark 5.13. In [17], the authors state that Proposition 5.12 applies to the group theoretic dual M^* , whereas their proof (given above) only applies to the Lie theoretic dual $M^{\tilde{*}}$. (See Section 2.2.) Through a personal conversation, Jon Carlson provided the following argument, which shows that Proposition 5.12 is actually independent of the choice of Hopf algebra structure on kE .

Let S be any subset of $J \setminus J^2$ and define $\overline{{}_S M}$ to be the submodule $\sum_{x \in S} \text{Ker}(x, M)$ of M . We then define $\overline{\mathfrak{K}}(M)$ to be the intersection of all submodules $\overline{{}_S M}$ for which the image of $S + J^2$ in $\mathbb{P}(J/J^2) \cong \mathbb{P}^1(k)$ is cofinite. It follows directly from the definitions that $\mathfrak{K}(M) \subseteq \overline{\mathfrak{K}}(M)$. Moreover, an argument similar to that given in the proof of Theorem 5.2 shows that $\overline{\mathfrak{K}}(M)$ has the constant image property. By Theorem 5.2 we then have $\overline{\mathfrak{K}}(M) \subseteq \mathfrak{K}(M)$, hence $\overline{\mathfrak{K}}(M) = \mathfrak{K}(M)$.

Proposition 5.14 (Carlson). *Let $\tau, \theta: kE \rightarrow kE$ be antipodes for any two Hopf algebra structures on kE . If M is a finite dimensional kE -module and M_τ^* and M_θ^* denote the kE -module structures on $\text{Hom}_k(M, k)$ induced by τ and θ , respectively, then $\mathfrak{K}(M_\tau^*) = \mathfrak{K}(M_\theta^*)$.*

PROOF. Let $x \in J$ and let α be the point in $\mathbb{A}^2(k)$ such that $x \equiv X_\alpha \pmod{J^2}$. By Lemma 5.4, we have

$$\tau(x) \equiv \theta(x) \equiv -X_\alpha \pmod{J^2}. \quad (5.2)$$

We may therefore write $\theta(x) = \tau(x) + y(x)$ with $y(x) \in J^2$.

Now choose $S \subseteq J \setminus J^2$ such that the image of $S + J^2$ in $\mathbb{P}(J/J^2)$ is cofinite, and such that $\mathfrak{K}(M_\theta^*) = \overline{{}_S M_\theta^*}$. It follows from the definitions of M_τ^* and M_θ^* that

$$\text{Ker}(x, M_\theta^*) = \text{Ker}(x + \tau(y(x)), M_\tau^*). \quad (5.3)$$

Letting $S' = \{x + \tau(y(x)) \mid x \in S\}$, note by (5.2) that the image of $S' + J^2$ in $\mathbb{P}(J/J^2)$ is equal to that of $S + J^2$, hence is cofinite. It now follows from (5.3) that

$$\mathfrak{K}(M_\theta^*) = \overline{{}_S M_\theta^*} = \overline{{}_{S'} M_\tau^*} \supseteq \mathfrak{K}(M_\tau^*).$$

A symmetric argument shows that we also have $\mathfrak{K}(M_\theta^*) \subseteq \mathfrak{K}(M_\tau^*)$. \square

Corollary 5.15. *If τ is the antipode for any Hopf algebra structure on kE and M is a finite dimensional kE -module, then*

$$\mathfrak{J}(M) \cong \mathfrak{K}(M_\tau^*)^\perp \quad \text{and} \quad \mathfrak{K}(M) \cong \mathfrak{J}(M_\tau^*)^\perp.$$

PROOF. This follows from Propositions 5.12 and 5.14 by setting $\theta = \tilde{\sigma}$. \square

5.5. The constant kernel property

The following definition can be thought of as being dual to that of the constant image property.

Definition 5.16 (Carlson, Friedlander and Suslin [17]). Given any rank r , a finite dimensional kE -module M has the *constant kernel property* if $\text{Ker}(X_\alpha, M)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$.

Proposition 5.17. *A kE -module M has the constant kernel property if and only if $M^{\tilde{*}}$ has the constant image property.*

PROOF. As in the proof of Proposition 5.12, we have $\text{Ker}(X_\alpha, M) = \text{Im}(X_\alpha, M^{\tilde{*}})^\perp$ for any non-zero $\alpha \in \mathbb{A}^r(k)$. Taking orthogonal complements shows that $\text{Ker}(X_\alpha, M)$ is independent of α if and only if $\text{Im}(X_\alpha, M^{\tilde{*}})$ is. \square

Corollary 5.18. *Let τ be the antipode for any Hopf algebra structure on kE . If M is a finite dimensional kE -module and M_τ^* denotes the dual kE -module structure induced from τ , then M has the constant kernel property if and only if M_τ^* has the constant image property.*

PROOF. By Propositions 5.17 and 4.2, we know that M has the constant kernel property if and only if $\text{Im}(x, M^{\tilde{*}})$ is independent of $x \in J \setminus J^2$. Choosing a non-zero $\alpha \in \mathbb{A}^r(k)$ and applying Lemma 5.4 to $\tilde{\sigma}$, we have

$$\tau(X_\alpha) \equiv \tilde{\sigma}(X_\alpha) = -X_\alpha \pmod{J^2}.$$

Writing $\tau(X_\alpha) = -X_\alpha + y$ with $y \in J^2$, we then have

$$\text{Im}(X_\alpha, M_\tau^*) = \text{Im}(X_\alpha + \tilde{\sigma}(y), M^{\tilde{*}}).$$

Because $X_\alpha + \tilde{\sigma}(y) \in J \setminus J^2$, this shows that M has the constant kernel property if and only if $\text{Im}(X_\alpha, M_\tau^*)$ is independent of α , i.e., if and only if M_τ^* has the constant image property. \square

The following two results appear as Lemma 8.5 and Theorem 8.6 of [17], respectively. Given our previous remarks, one readily verifies that their proofs have no dependence on the choice of Hopf algebra structure on kE . We therefore use the group theoretic antipode σ without any loss of generality.

Lemma 5.19. *If M is a kE -module of constant rank, then*

$$\text{Ker}(X_\alpha, M/J\mathfrak{K}(M)) = \mathfrak{K}(M)/J\mathfrak{K}(M).$$

In particular, $M/J\mathfrak{K}(M)$ has the constant kernel property, that is, $(\mathfrak{K}(M)/J\mathfrak{K}(M))^$ has the constant image property.*

PROOF. If $m \in \mathfrak{K}(M)$, then we clearly have $X_\alpha m \in J\mathfrak{K}(M)$, hence

$$\mathfrak{K}(M)/J\mathfrak{K}(M) \subseteq \text{Ker}(X_\alpha, M/J\mathfrak{K}(M)).$$

Conversely, if $m \in M$ and $X_\alpha m \in J\mathfrak{K}(M)$, then it follows by Theorem 5.2 and the constant image property that there exists $m' \in \mathfrak{K}(M)$ such that $X_\alpha m' = X_\alpha m$. We then have $m - m' \in \text{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$ by Theorem 5.7, forcing $m \in \mathfrak{K}(M)$. This shows that $\text{Ker}(X_\alpha, M/J\mathfrak{K}(M)) \subseteq \mathfrak{K}(M)/J\mathfrak{K}(M)$, proving the first statement. The second follows from Corollary 5.18. \square

Theorem 5.20. *If M is a finite dimensional kE -module of constant rank, then $\mathfrak{J}(M) = J\mathfrak{K}(M)$.*

PROOF. Since $\mathfrak{K}(M)$ has the constant image property, $\text{Im}(X_\alpha, \mathfrak{K}(M)) = J\mathfrak{K}(M)$ for all non-zero $\alpha \in \mathbb{A}^2(k)$. Letting $S \subseteq \mathbb{P}^1(k)$ be cofinite such that $M_S = \mathfrak{J}(M)$, we thus have $\mathfrak{J}(M) = \sum_{\bar{\alpha} \in S} \text{Im}(X_\alpha, M) \supseteq \sum_{\bar{\alpha} \in S} \text{Im}(X_\alpha, \mathfrak{K}(M)) = J\mathfrak{K}(M)$. To prove the reverse containment, note that $(M/J\mathfrak{K}(M))^* \subseteq \mathfrak{K}(M^*)$ by Lemma 5.19 and Theorem 5.2. Applying Corollary 5.15 now yields

$$\mathfrak{J}(M) = (\mathfrak{K}(M^*))^\perp \subseteq ((M/J\mathfrak{K}(M))^*)^\perp = J\mathfrak{K}(M). \quad \square$$

5.6. Duality for subquotients in the generic kernel filtration

In this section we introduce a duality formula that will be used to study the generic kernel filtration (5.1) and Question 5.10 in terms of the generic kernels of both M and M^* . As before, our approach will illustrate that the following results are independent of the choice of Hopf algebra structure on kE .

Lemma 5.21. *Let τ be the antipode for any Hopf algebra structure on kE . If M is a kE -module of constant rank and M_τ^* denotes the dual kE -module structure induced by τ , then $\mathfrak{K}(M_\tau^*) \cong (M/J\mathfrak{K}(M))_\tau^*$.*

PROOF. By Corollary 5.15 and Theorem 5.20 we have

$$\mathfrak{K}(M_\tau^*) = (\mathfrak{J}(M))^\perp = (J\mathfrak{K}(M))^\perp \cong (M/J\mathfrak{K}(M))_\tau^*. \quad \square$$

Lemma 5.22. *Let τ be the antipode for any Hopf algebra structure on kE . If M is a kE -module in any rank r and N is a submodule of M , then*

- (i) $J^{-1}(N^\perp) = (JN)^\perp$ and
- (ii) $J(N^\perp) = (J^{-1}N)^\perp$,

where the action of kE on $\text{Hom}_k(M, k)$ is that induced by τ .

PROOF. To prove (i), suppose $f \in J^{-1}(N^\perp)$. For any $n \in JN$, we may write $n = X_1n_1 + \cdots + X_rn_r$ with $n_i \in N$. We then have

$$f(n) = f(X_1n_1) + \cdots + f(X_rn_r) = (\tau(X_1).f)(n_1) + \cdots + (\tau(X_r).f)(n_r) = 0$$

since each $\tau(X_i)$ lies in J . This shows that $f \in (JN)^\perp$. Conversely, if $f \in (JN)^\perp$ and $n \in N$, then

$$(X_i.f)(n) = f(\tau(X_i)n) = 0 \quad \text{for all } 1 \leq i \leq r$$

since $\tau(X_i)n \in JN$. It follows that $X_i.f \in N^\perp$ for all i , that is, $J.f \subseteq N^\perp$.

To prove (ii), note that

$$J^{-1}(N) = J^{-1}((N^\perp)^\perp) = (J(N^\perp))^\perp$$

by (i) so that $(J^{-1}N)^\perp = J(N^\perp)$. □

We are now in a position to prove the main theorem of this section.

Theorem 5.23. *Let τ be the antipode for any Hopf algebra structure on kE . If M is a kE -module of constant rank, then for all $a, b \in \mathbb{Z}$ with $a \leq b$ we have*

$$J^a \mathfrak{K}(M_\tau^*) / J^b \mathfrak{K}(M_\tau^*) \cong (J^{-b+1} \mathfrak{K}(M) / J^{-a+1} \mathfrak{K}(M))_\tau^*.$$

PROOF. We first need to prove that $J^{-a} J \mathfrak{K}(M) = J^{-a+1} \mathfrak{K}(M)$ for all $a > 0$. For the leftwards containment, note that if $m \in M$ and $J^{a-1}m \in \mathfrak{K}(M)$, then we clearly have $J^a m \subseteq J \mathfrak{K}(M)$. Conversely, suppose that $m \in M$ satisfies $J^a m \subseteq J \mathfrak{K}(M)$. For any $y \in J^{a-1}$, we then have $X_1 y m \in J \mathfrak{K}(M)$. Because $\mathfrak{K}(M)$ has the constant image property, there exists $m' \in \mathfrak{K}(M)$ such that $X_1 m' = X_1 y m$. It follows by Theorem 5.7 that $m' - y m \in \mathfrak{K}(M)$, hence $y m \in \mathfrak{K}(M)$. This shows that $J^{a-1} m \subseteq \mathfrak{K}(M)$.

Having established the above equality, it follows by Lemmas 5.21 and 5.22 that

$$\frac{J^a \mathfrak{K}(M_\tau^*)}{J^b \mathfrak{K}(M_\tau^*)} \cong \frac{J^a (J \mathfrak{K}(M))^\perp}{J^b (J \mathfrak{K}(M))^\perp} = \frac{(J^{-a+1} \mathfrak{K}(M))^\perp}{(J^{-b+1} \mathfrak{K}(M))^\perp} \cong \left(\frac{J^{-b+1} \mathfrak{K}(M)}{J^{-a+1} \mathfrak{K}(M)} \right)_\tau^*. \quad \square$$

5.7. Blocks of small length in certain subquotients

When proving that Question 5.10 has an affirmative answer in the case $p = 3$, it will be important to know that if M is a kE -module of constant Jordan type, then the number of blocks of length one in the action of X_α on $J^{-1} \mathfrak{K}(M)$ is the same for all non-zero $\alpha \in \mathbb{A}^2(k)$. In this section we prove a more general statement about blocks having small length in the action of X_α on the subquotients $J^{-i} \mathfrak{K}(M) / J^j \mathfrak{K}(M)$ for $i \geq 1, j \geq 2$. We begin with a key lemma.

Lemma 5.24. *If M is a kE -module of constant rank, then for all non-zero $\alpha \in \mathbb{A}^2(k)$ and all $0 \leq n \leq i$ we have*

$$\text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^n, M) = \text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \cap \text{Im}(X_\alpha^n, J^{-i}\mathfrak{K}(M)).$$

PROOF. The leftwards containment is clear, so let $m \in \text{Ker}(X_\alpha, M) \cap \text{Im}(X_\alpha^n, M)$. Note that $\text{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$ by Theorem 5.7. It follows that

$$\text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \subseteq \text{Ker}(X_\alpha, M) \subseteq \text{Ker}(X_\alpha, \mathfrak{K}(M)) \subseteq \text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)),$$

whence equality holds throughout. In particular, we have $m \in \text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M))$. Also, there exists $m' \in M$ such that $X_\alpha^n m' = m$. Since $m \in \mathfrak{K}(M)$, Proposition 5.8 implies $m' \in X_\alpha^{-n}\mathfrak{K}(M) = J^{-n}\mathfrak{K}(M) \subseteq J^{-i}\mathfrak{K}(M)$, hence $m \in \text{Im}(X_\alpha^n, J^{-i}\mathfrak{K}(M))$. \square

Theorem 5.25. *Let M be a kE -module of constant rank and let $i \geq 1$, $j \geq 2$. If $n \leq \min\{i, j-1\}$, then the number of Jordan blocks of length n in the action of X_α on $J^{-i}\mathfrak{K}(M)/J^j\mathfrak{K}(M)$ is equal to the number of Jordan blocks of length n in the action of X_α on M .*

PROOF. It suffices to show that

$$\frac{\text{Ker}(X_\alpha, M) \cap X_\alpha^{n-1}M}{\text{Ker}(X_\alpha, M) \cap X_\alpha^n M} \cong \frac{\text{Ker}\left(X_\alpha, \frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)}\right) \cap X_\alpha^{n-1}\frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)}}{\text{Ker}\left(X_\alpha, \frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)}\right) \cap X_\alpha^n\frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)}} \quad (5.4)$$

since the dimension of the left hand side is equal to the number of Jordan blocks of length n in the action of X_α on M , and the dimension of the right hand side is equal to the number of Jordan blocks of length n in the action of X_α on $J^{-i}\mathfrak{K}(M)/J^j\mathfrak{K}(M)$.

Note by Lemma 5.24 that we have

$$\frac{\text{Ker}(X_\alpha, M) \cap X_\alpha^{n-1}M}{\text{Ker}(X_\alpha, M) \cap X_\alpha^n M} = \frac{\text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \cap X_\alpha^{n-1}J^{-i}\mathfrak{K}(M)}{\text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \cap X_\alpha^n J^{-i}\mathfrak{K}(M)}.$$

By the second isomorphism theorem we also have

$$\frac{\text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \cap X_\alpha^{n-1}J^{-i}\mathfrak{K}(M)}{\text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \cap X_\alpha^n J^{-i}\mathfrak{K}(M)} \cong \frac{(\text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \cap X_\alpha^{n-1}J^{-i}\mathfrak{K}(M)) + X_\alpha^n J^{-i}\mathfrak{K}(M)}{X_\alpha^n J^{-i}\mathfrak{K}(M)}.$$

Consider the natural injective k -linear map

$$\frac{(\text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \cap X_\alpha^{n-1}J^{-i}\mathfrak{K}(M)) + X_\alpha^n J^{-i}\mathfrak{K}(M)}{X_\alpha^n J^{-i}\mathfrak{K}(M)} \longrightarrow \frac{\left(\text{Ker} \left(X_\alpha, \frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)} \right) \cap X_\alpha^{n-1} \frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)} \right) + X_\alpha^n \frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)}}{X_\alpha^n \frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)}},$$

where the latter term is isomorphic to the right hand side of (5.4). In order to prove the theorem, we need to show that this map is also surjective.

Let $m \in J^{-i}\mathfrak{K}(M)$ such that

$$m + J^j\mathfrak{K}(M) \in \text{Ker} \left(X_\alpha, \frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)} \right) \cap X_\alpha^{n-1} \frac{J^{-i}\mathfrak{K}(M)}{J^j\mathfrak{K}(M)}.$$

Because $X_\alpha m \in J^j\mathfrak{K}(M)$, it follows from Theorem 5.2 and Proposition 4.4 that there exists $m' \in \mathfrak{K}(M)$ such that $X_\alpha m = X_\alpha^j m'$. We then have $X_\alpha(m - X_\alpha^{j-1}m') = 0$. Note by our choice of m that there also exists $m'' \in J^{-i}\mathfrak{K}(M)$ such that

$$m - X_\alpha^{n-1}m'' \in J^j\mathfrak{K}(M) = X_\alpha^j\mathfrak{K}(M),$$

hence $m \in X_\alpha^{n-1}J^{-i}\mathfrak{K}(M)$. Finally, we have $X_\alpha^{j-1}m' \in X_\alpha^n\mathfrak{K}(M) \subseteq X_\alpha^n J^{-i}\mathfrak{K}(M)$ since $n \leq j - 1$. Combining the above information now yields

$$m = (m - X_\alpha^{j-1}m') + X_\alpha^{j-1}m' \in (\text{Ker}(X_\alpha, J^{-i}\mathfrak{K}(M)) \cap X_\alpha^{n-1}J^{-i}\mathfrak{K}(M)) + X_\alpha^n J^{-i}\mathfrak{K}(M)$$

as required. \square

We now show that Theorem 5.25 allows us to generalise Lemma 3.3 of Benson [6]. The main distinction here is that the latter result only applies to kE -modules of constant Jordan type containing no blocks of length one.

Theorem 5.26. *If M is any kE -module of constant Jordan type, then the subquotient $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ also has constant Jordan type.*

PROOF. Fix a non-zero $\alpha \in \mathbb{A}^2(k)$ and let $[3]^a[2]^b[1]^c$ be the Jordan type of X_α on $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. Note that the total number of Jordan blocks in the action of X_α on M is independent of α and equal to $\dim_k \text{Ker}(X_\alpha, M)$, which in turn is equal to $\dim_k \text{Ker}(X_\alpha, J^{-1}\mathfrak{K}(M))$ by Theorem 5.7. We also have

$$\begin{aligned} \dim_k \text{Ker}(X_\alpha, J^{-1}\mathfrak{K}(M)) &= \dim_k J^{-1}\mathfrak{K}(M)/X_\alpha J^{-1}\mathfrak{K}(M) \\ &= \dim_k \frac{J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)}{X_\alpha J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)}. \end{aligned}$$

The latter is the total number of Jordan blocks of X_α acting on $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. This shows that $a + b + c$ is independent of α . We also have

$$3a + 2b + c = \dim_k J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M),$$

which is clearly independent of α . Finally, Theorem 5.25 implies that c is equal to the number of Jordan blocks of length one in the action of X_α on M , which is also independent of α . The fact that $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ has constant Jordan type now follows from the non-singularity of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{Z}). \quad \square$$

5.8. Question 5.10 in the cases $p = 3$ and $J^2\mathfrak{K}(M) = 0$

In this section we identify two special cases in which Question 5.10 has an affirmative answer.

Theorem 5.27. *If $p = 3$ and M is a kE -module of constant Jordan type, then $J^{-i}\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$.*

PROOF. If M is a $k(\mathbb{Z}/3)^2$ -module of constant Jordan type, then $M = J^{-2}\mathfrak{K}(M)$ by Corollary 5.9. Hence we need only check that $J^{-1}\mathfrak{K}(M)$ has constant Jordan type. This involves essentially the same argument as given in the proof of Theorem 5.26. We again fix a non-zero $\alpha \in \mathbb{A}^2(k)$ and let $[3]^a[2]^b[1]^c$ be the Jordan type of X_α on $J^{-1}\mathfrak{K}(M)$. Theorem 5.7 implies that the total number of Jordan blocks of X_α acting on $J^{-1}\mathfrak{K}(M)$ is equal to that of X_α on M , hence $a + b + c$ is independent of α . We have $3a + 2b + c = \dim_k J^{-1}\mathfrak{K}(M)$ independent of α , and Theorem 5.25 tells us that c is equal to the number of Jordan blocks of length one in the action of X_α on M , hence is independent of α . The fact that $J^{-1}\mathfrak{K}(M)$ has constant Jordan type again follows from the non-singularity of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{Z}). \quad \square$$

In order to address the case in which $J^2\mathfrak{K}(M) = 0$, we make the following observation. Let M be any kE -module and let $n \geq 0$. For each non-zero $\alpha \in \mathbb{A}^2(k)$ there is a natural short exact sequence

$$0 \longrightarrow L_{n,\alpha} \longrightarrow \text{Im}(X_\alpha^n, M/J^{j+1}\mathfrak{K}(M)) \longrightarrow \text{Im}(X_\alpha^n, M/J^j\mathfrak{K}(M)) \longrightarrow 0 \quad (5.5)$$

where

$$L_{n,\alpha} = \text{Im}(X_\alpha^n, M/J^{j+1}\mathfrak{K}(M)) \cap J^j\mathfrak{K}(M)/J^{j+1}\mathfrak{K}(M).$$

If $M/J^{j+1}\mathfrak{K}(M)$ has constant Jordan type, then the dimension of the middle term is independent of α . We now introduce a criterion under which the dimensions of the outer two terms would also be independent of α .

Lemma 5.28. *Suppose $M/J^{j+1}\mathfrak{K}(M)$ has constant Jordan type and fix $n \geq 0$. If there exists a submodule N of $M/J^{j+1}\mathfrak{K}(M)$ such that $L_{n,\alpha} = \text{Im}(X_\alpha^n, N)$ for all non-zero $\alpha \in \mathbb{A}^2(k)$, then $\text{rank}(X_\alpha^n, M/J^j\mathfrak{K}(M))$ is independent of α .*

PROOF. Since $M/J^{j+1}\mathfrak{K}(M)$ has constant Jordan type, we see from the sequence (5.5) that

$$\text{rank}(X_\alpha^n, N) + \text{rank}(X_\alpha^n, M/J^j\mathfrak{K}(M))$$

is constant as a function of α . Following the proof of Proposition 2.2, there exists a dense open subset $U \subseteq \mathbb{A}^2(k)$ such that $\alpha \in U$ if and only if $\text{rank}(X_\alpha^n, M/J^j\mathfrak{K}(M))$ is maximal. Similarly, there exists a dense open subset $V \subseteq \mathbb{A}^2(k)$ such that $\alpha \in V$ if and only if $\text{rank}(X_\alpha^n, N)$ is maximal. Since $\mathbb{A}^2(k)$ is irreducible, $U \cap V$ is non-empty. Choosing $\alpha \in U \cap V$, we simultaneously have $\text{rank}(X_\alpha^n, M/J^j\mathfrak{K}(M))$ maximal and $\text{rank}(X_\alpha^n, N)$ maximal. This forces both summands to be constant in α . \square

We now use the above theory to show that Question 5.10 is true in the special case where $J^2\mathfrak{K}(M) = 0$. We make use of the group theoretic dual M^* without any loss of generality, because the proof is independent of the choice of Hopf algebra structure on kE .

Theorem 5.29. *If M is a kE -module of constant Jordan type such that $J^2\mathfrak{K}(M) = 0$, then $J^{-i}\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$.*

PROOF. Since $J^2\mathfrak{K}(M) = 0$, we have $M^* = J^{-1}\mathfrak{K}(M^*)$ by Theorem 5.23. We first show that $M^*/J^j\mathfrak{K}(M^*)$ has constant Jordan type for all $j \geq 0$ using downwards induction on j , beginning at $M^* = M^*/J^0\mathfrak{K}(M^*)$. So assume that $M^*/J^{j+1}\mathfrak{K}(M^*)$ has constant Jordan type and note for a non-zero $\alpha \in \mathbb{A}^2(k)$ that any Jordan block in the action of X_α on $M^*/J^j\mathfrak{K}(M^*)$ has length at most $j+1$. If $n \leq j$, let $m \in M^*/J^{j+1}\mathfrak{K}(M^*)$ such that $X_\alpha^n m \in J^j\mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)$. Since $\mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)$ has the constant image property, there exists $m' \in \mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)$ such that $X_\alpha^n m' = X_\alpha^n m$. Applying the sequence (5.5) to M^* , this yields

$$L_{n,\alpha} = \text{Im}(X_\alpha^n, \mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)) \quad \text{for all non-zero } \alpha \in \mathbb{A}^2(k).$$

It follows from Lemma 5.28 that $\text{rank}(X_\alpha^n, M^*/J^j\mathfrak{K}(M^*))$ is independent of α for all $n \leq j$. If $[1]^{a_1} \dots [j+1]^{a_{j+1}}$ is the Jordan type of X_α on $\mathfrak{K}(M^*)/J^{j+1}\mathfrak{K}(M^*)$, then we have

$$\begin{aligned} a_1 + 2a_2 + 3a_3 + \dots + ja_j + (j+1)a_{j+1} &= \dim_k M^*/J^j\mathfrak{K}(M^*) \\ a_2 + 2a_3 + \dots + (j-1)a_j + ja_{j+1} &= \text{rank}(X_\alpha, M^*/J^j\mathfrak{K}(M^*)) \\ &\vdots \\ a_j + 2a_{j+1} &= \text{rank}(X_\alpha^{j-1}, M^*/J^j\mathfrak{K}(M^*)) \\ a_{j+1} &= \text{rank}(X_\alpha^j, M^*/J^j\mathfrak{K}(M^*)). \end{aligned}$$

Because the terms on the right are independent of α , these equations have a solution for (a_1, \dots, a_{j+1}) that is also independent of α . This shows that $M^*/J^j\mathfrak{K}(M^*)$ has constant Jordan type.

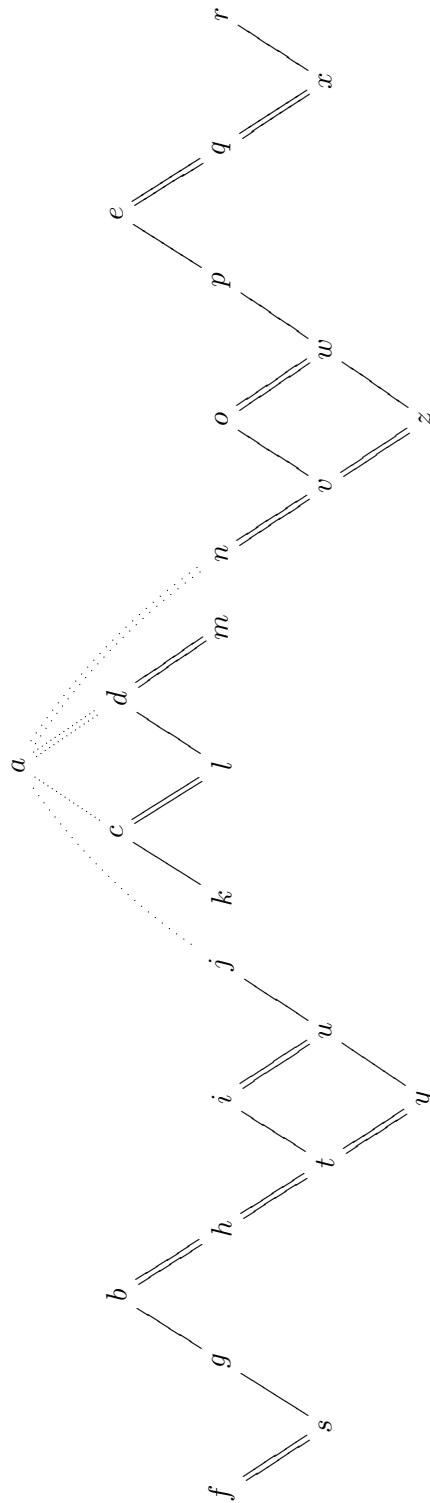
Having proved the claim, setting $j = -i + 1$, it now follows by another application of Theorem 5.23 that $J^{-i}\mathfrak{K}(M)$ has constant Jordan type for all $i \geq 0$ as required. \square

5.9. Some counterexamples to Question 5.10

We now present an example of a kE -module M of constant Jordan type such that $J^{-1}\mathfrak{K}(M)$ does not have constant Jordan type.

Example 5.30. Let M be the 26 dimensional kE -module having basis

$$a, b, c, \dots, x, y, z$$



We first prove that M has constant Jordan type by verifying that $\text{rank}(X_\alpha^n, M)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^2(k)$ for $n = 1, 2, 3$. By inspection of the diagram we see that $\text{rank}(X_2, M) = 13$. For any $\lambda \in k$, the image of $X_1 + \lambda X_2$ on M is spanned by the elements

$$c + j + \lambda(d + n), g + \lambda h, k + \lambda l, l + \lambda m, p + \lambda q, s, t, u, v, w, x, y, z,$$

which are readily confirmed to be linearly independent. Hence $\text{rank}(X_1 + \lambda X_2, M) = 13$ as well. Next observe that $\text{rank}(X_2^2, M) = 5$, and for any $\lambda \in k$, the image of $(X_1 + \lambda X_2)^2$ on M is spanned by the linearly independent elements

$$k + u + 2\lambda l + \lambda^2(m + v), s + \lambda^2 t, w + \lambda^2 x, y, z$$

so that $\text{rank}((X_1 + \lambda X_2)^2, M) = 5$. Finally, we have $\text{rank}(X_2^3, M) = 2$, and for any $\lambda \in k$, the image of $(X_1 + \lambda X_2)^3$ on M is the submodule spanned by

$$\lambda^3 y, y + \lambda^3 z, z,$$

which in any case has dimension two. Thus $\text{rank}((X_1 + \lambda X_2)^3, M) = 2$ as required. This shows that M has constant Jordan type $[4]^2[3][2]^5[1]^5$.

To see that $J^{-1}\mathfrak{K}(M)$ does not have constant Jordan type, note that $\mathfrak{K}(M)$ is the submodule of M generated by

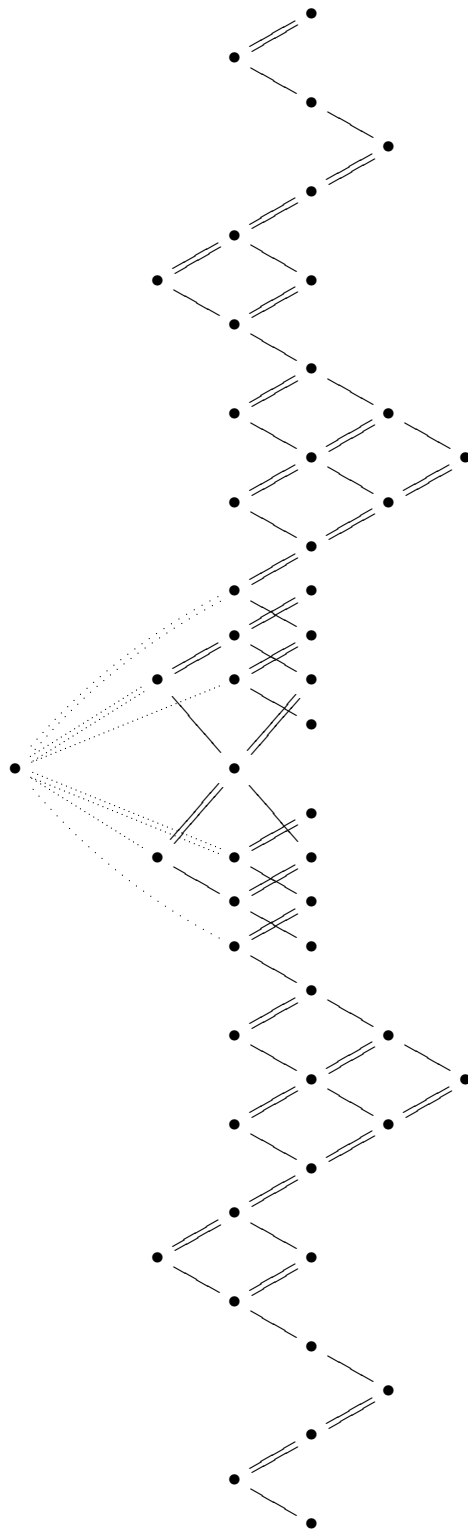
$$f, g, h, i, j, k, l, m, n, o, p, q, r$$

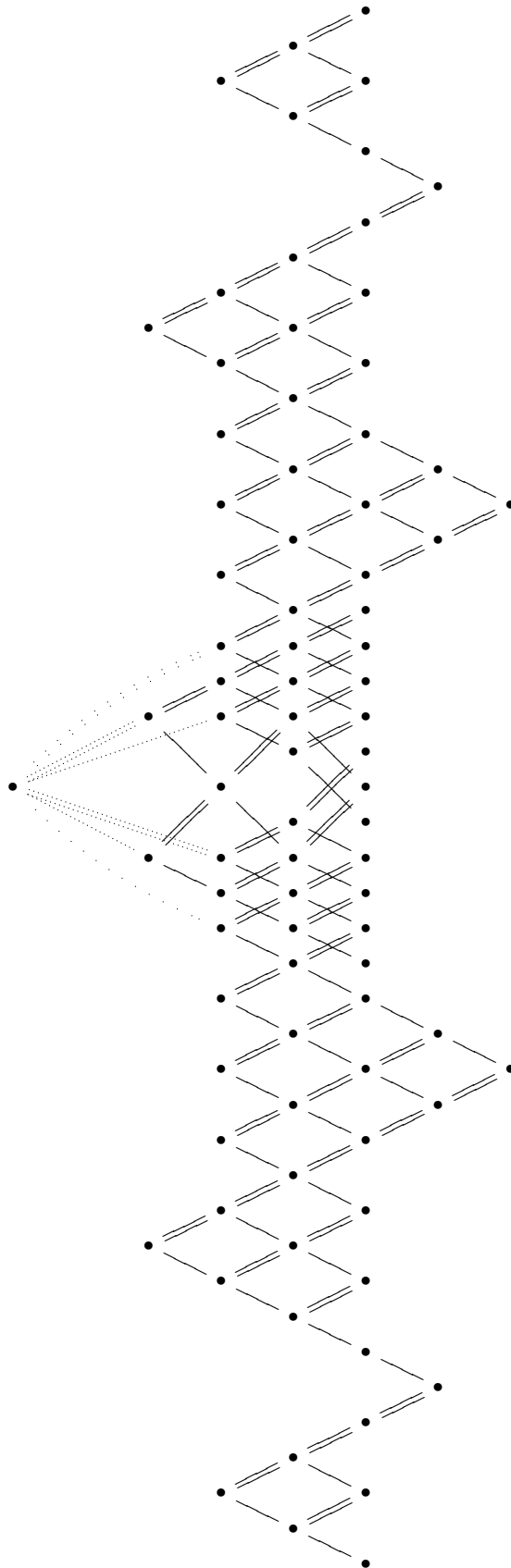
since this is the largest submodule of M having the constant image property. It follows that $J^{-1}\mathfrak{K}(M)$ is the submodule of M generated by

$$f, b, i, j, c, d, n, o, e, r.$$

Observe that $J^{-1}\mathfrak{K}(M)$ has a direct summand N generated by the elements f, b, i, j . Clearly N cannot have constant Jordan type because, for example, $X_2^3 N = \text{span}_k(y)$, whereas $X_1^3 N = 0$. By Theorem 2.18 (ii), this shows that $J^{-1}\mathfrak{K}(M)$ cannot have constant Jordan type.

While Example 5.30 does answer Question 5.10 in the negative, one might be tempted to ask whether or not there exists a ‘magic number’ $i > 1$ for which $J^{-i}\mathfrak{K}(M)$ has constant Jordan type whenever M does. We now present a family of counterexamples to this question. The first diagram represents a kE -module of constant Jordan type satisfying $J^3\mathfrak{K}(M) = 0$ for which $J^{-2}\mathfrak{K}(M)$ does not have constant Jordan type. The second represents such a module for which $J^{-3}\mathfrak{K}(M)$ does not have constant Jordan type. Higher counterexamples may be constructed similarly by following the inherent pattern.





CHAPTER 6

On the structure of the subquotient $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$

In this chapter we examine a theorem of Benson related to Rickard's Conjecture 2.25. The proof of that theorem involves studying the subquotient $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ of a kE -module M of constant Jordan type containing no blocks of length one. Although we come nowhere near obtaining a full structure theorem for these subquotients, we do prove that they contain certain 'diamond shaped' submodules isomorphic to those of type $D_pVD_pV \dots VD_p$ described in Section 2.7.

6.1. Benson's theorem on Rickard's conjecture

The following theorem is the main result of Benson [6]. It proves the special case of Rickard's Conjecture 2.25 in which $a_1 = 0$.

Theorem 6.1. *If $r \geq 2$ and M is a kE -module of constant Jordan type that contains no blocks of length one, then the total number of Jordan blocks of M is divisible by p .*

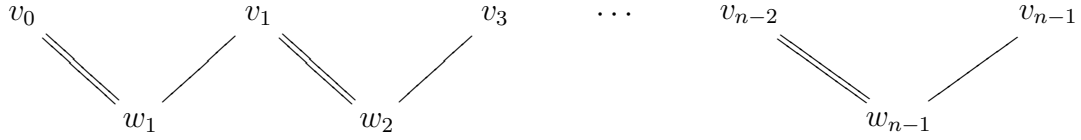
PROOF. It suffices to restrict to the rank two case since the restriction of M to any rank two shifted subgroup of kE^\times will be a module for that subgroup having the same constant Jordan type.

Now consider the subquotient $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. It follows from Theorem 5.26 and its proof that this is also a module of constant Jordan type containing no blocks of length one. Furthermore, the same proof shows that the number of Jordan blocks of $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ is equal to the number of Jordan blocks of M , which in turn is equal to the number of Jordan blocks of $\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. The latter quantity is equal to $\dim_k \mathfrak{K}(M)/J\mathfrak{K}(M)$. It therefore suffices to assume that $M = J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ and to prove for every direct summand N of $\mathfrak{K}(M)$ that the dimension of N/JN is divisible by p .

So let N be an indecomposable direct summand of $\mathfrak{K}(M)$. Note that N is also an indecomposable module for kE/J^2 , and all such modules correspond to representations of the Kronecker quiver. Since $\mathfrak{K}(M)$ has the constant image property, the map

$$X_\alpha: \mathfrak{K}(M)/J\mathfrak{K}(M) \longrightarrow J\mathfrak{K}(M)$$

is surjective for any non-zero $\alpha \in \mathbb{A}^2(k)$. By the Kronecker classification, this means that N is isomorphic to a W -module of radical length two, hence the structure of N is given by the module diagram



as described in Section 4.3. Note that we have altered the labeling of the top vertices for the sake of this proof.

Since M contains no Jordan blocks of length one, we have $\text{Ker}(X_\alpha, M) \subseteq \text{Im}(X_\alpha, M)$ for all non-zero $\alpha \in \mathbb{A}^2(k)$. It follows from Proposition 2.10 that this remains true after extending k to the field $K = k[[t]][t^{-1}]$ of Laurent power series. The kernel of the element $X_1 + tX_2 \in kE \otimes_k K$ on $N \otimes_k K$ is one dimensional, spanned by

$$v_0 - tv_1 + t^2v^2 - \cdots \pm t^{n-1}v_{n-1}.$$

Hence there exists an element $\sum_i (-1)^i t^i u_i \in M \otimes_k K$ such that

$$(X_1 + tX_2) \sum_i (-1)^i t^i u_i = \sum_{j=0}^{n-1} (-1)^j t^j v_j. \tag{6.1}$$

Observe for any non-zero $\alpha \in \mathbb{A}^2(k)$ that the linear map

$$X_\alpha: M/\mathfrak{K}(M) \longrightarrow \mathfrak{K}(M)/J\mathfrak{K}(M)$$

is injective by Theorem 5.7. In particular, because X_1 and X_2 are injective, we have $u_i = 0$ for all $i < 0$.

We also claim that $u_i = 0$ for all $i \geq n-1$. Suppose first that $u_{n-1} \neq 0$. Let $m > n-1$ be the smallest integer such that there exists a linear relation $\sum_{i=n-1}^m \lambda_i u_i = 0$. Such an integer exists because M is finite dimensional. Then u_{n-1}, \dots, u_{m-1} are linearly independent and u_m is a linear combination of them. It follows from (6.1) that

$$\begin{aligned}
 X_2 u_m &\in \text{span}_k(X_2 u_{n-1}, \dots, X_2 u_{m-1}) = \text{span}_k(X_1 u_n, \dots, X_1 u_m) \\
 &\subseteq \text{span}_k(X_1 u_{n-1}, \dots, X_1 u_{m-1}).
 \end{aligned}$$

This shows for any non-zero $\alpha \in \mathbb{A}^2(k)$ that we have an injective map

$$X_\alpha: \text{span}_k(u_{n-1}, \dots, u_{m-1}) \longrightarrow \text{span}_k(X_1 u_{n-1}, \dots, X_1 u_{m-1}),$$

which is necessarily an isomorphism. Letting X'_1 denote the linear map inverse to X_1 , the composition

$$X'_1 \circ (\lambda X_1 - X_2) = \lambda \text{Id} - (X'_1 \circ X_2)$$

is also an isomorphism for all $\lambda \in k$. In particular, its injectivity implies that $X'_1 \circ X_2$ has no non-zero eigenvector, contradicting the fact that k is algebraically closed. This shows that $u_{n-1} = 0$. The relations $X_1 u_{i+1} + X_2 u_i = 0$ for all $i \geq n - 1$ then show that $u_i = 0$ for all $i \geq n - 1$.

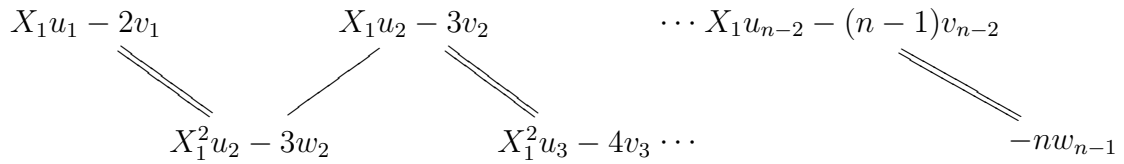
The above remarks applied to (6.1) now yield the equations

$$\begin{aligned} X_1 u_0 &= v_0 \\ X_1 u_1 - X_2 u_0 &= v_1 \\ &\vdots \\ X_1 u_{n-2} - X_2 u_{n-3} &= v_{n-2} \\ -X_2 u_{n-2} &= v_{n-1}. \end{aligned} \tag{6.2}$$

By the relations defining the W -module structure of N , it follows that

$$\begin{aligned} X_1(X_1 u_1 - 2v_1) &= X_1(X_2 u_0 - v_1) = X_2 v_0 - X_1 v_1 = w_1 - w_1 = 0, \\ X_2(X_1 u_{i-1} - i v_{i-1}) &= X_1^2 u_i - (i+1)w_i = X_1(X_1 u_i - (i+1)v_i) \quad 2 \leq i \leq n-2, \\ X_2(X_1 u_{n-2} - (n-1)v_{n-2}) &= -n w_{n-1}. \end{aligned}$$

This shows that there exists a submodule of $\mathfrak{K}(M)$ whose structure is indicated in the following diagram.



Note that the elements represented by vertices in the above graph are not necessarily linearly independent, or even non-zero. Nonetheless, we have

$$-n w_{n-1} \in (X_2 X_1^{-1})^{n-2} \{0\}$$

where, for a subspace V of $\mathfrak{K}(M)$, $X_1^{-1}V$ denotes the set of $m \in \mathfrak{K}(M)$ for which $X_1 m \in V$. Because N is indecomposable, it follows from the W -module structure of N that $w_{n-1} \notin (X_2 X_1^{-1})^{n-2} \{0\}$, hence we must have $-n = 0$ in k . This gives us

$$\dim_k N/JN = n \equiv 0 \pmod{p}$$

as required. □

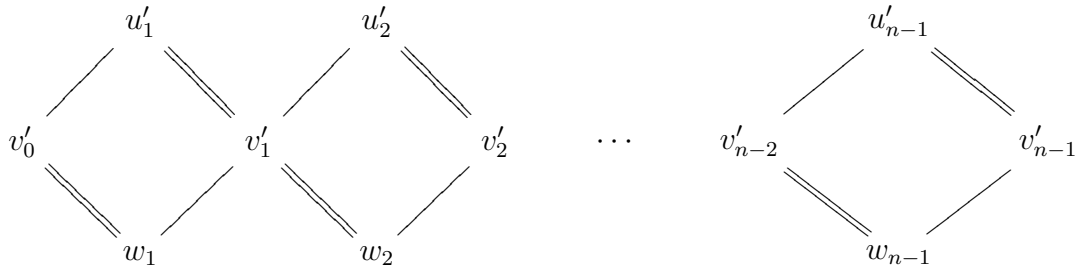
6.2. Diamond shaped submodules of $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$

Returning to the rank two case, we are now in a position to prove the main theorem of the chapter.

Theorem 6.2. *Let M be a kE -module of constant Jordan type containing no blocks of length one and let $N \cong W_{n,2}$ be a direct summand of $\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ with n minimal. Label the top and bottom vertices of N by v_0, \dots, v_{n-1} and w_1, \dots, w_{n-1} , respectively, as in the proof of Theorem 6.1. Then there exist elements $v'_i \in \mathfrak{K}(M)/J^2\mathfrak{K}(M)$ for $0 \leq i \leq n-1$, and $u'_i \in J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ for $1 \leq i \leq n-1$ with $p \nmid i$ such that*

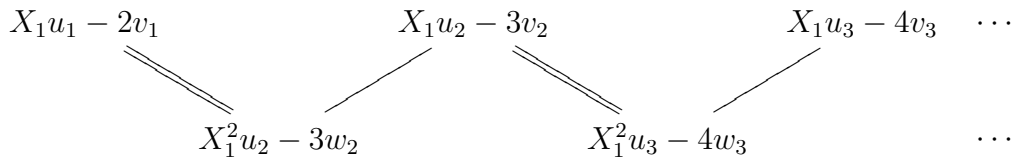
- (i) $v'_i \equiv v_i \pmod{J\mathfrak{K}(M)/J^2\mathfrak{K}(M)}$ for all $0 \leq i \leq n-1$, and
- (ii) $X_1u'_i = v'_{i-1}$ and $X_2u'_i = v'_i$ for all $1 \leq i \leq n-1$ with $p \nmid i$.

In other words, we obtain a submodule of $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ having diagram



where u'_i exists if $p \nmid i$.

PROOF. Consider the submodule L of N constructed in the proof of Theorem 6.1 whose structure is illustrated in the following diagram.



We claim that the top vertices of L lie in $J\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. We first write

$$\mathfrak{K}(M)/J^2\mathfrak{K}(M) \cong \bigoplus_j W_{n_j,2}$$

by the Kronecker classification and let $\phi_j: L \rightarrow W_{n_j,2}$ be the composition of the inclusion of L in $\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ with the projection of $\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ onto $W_{n_j,2}$. Fixing j , we label the top and bottom vertices of $W_{n_j,2}$ by a_0, \dots, a_{n_j-1} and b_1, \dots, b_{n_j-1} , respectively. Because $X_1(X_1u_1 - 2v_1) = 0$ in L , we have

$$\phi_j(X_1u_1 - 2v_1) \equiv \lambda_0 a_0 \pmod{\text{Soc}(W_{n_j,2})}$$

for some $\lambda_0 \in k$. We then have $X_1\phi_j(X_1u_2 - 3v_2) = X_2\phi_j(X_1u_1 - 2v_1) = \lambda_0b_1$ so that $\phi_j(X_1u_2 - 3v_2) \equiv \lambda_1a_0 + \lambda_0a_1 \pmod{\text{Soc}(W_{n_j,2})}$ for some $\lambda_1 \in k$. Continuing in this way, we have

$$\phi_j(X_1u_i - (i+1)v_i) \equiv \sum_{l=0}^{i-1} \lambda_{i-l-1}a_l \pmod{\text{Soc}(W_{n_j,2})} \quad \text{for } 1 \leq i \leq n-2.$$

We also have $X_2\phi_j(X_1u_{n-2} - (n-1)v_{n-2}) = 0$, hence

$$0 = \sum_{l=0}^{n-3} \lambda_{i-l-1}X_2a_l = \sum_{l=0}^{n-3} \lambda_{i-l-1}b_{l+1}.$$

Because n was assumed to be minimal, $b_{n-2} \neq 0$. It follows that each $\lambda_i = 0$ so that $\phi_j(X_1u_i - (i+1)v_i) \in \text{Soc}(W_{n_j,2})$. But if $m \in L$, then $m = \sum_j \phi_j(m)$, thus

$$X_1u_i - (i+1)v_i \in \text{Soc}(\mathfrak{K}(M)/J^2\mathfrak{K}(M)) = J\mathfrak{K}(M)/J^2\mathfrak{K}(M)$$

for all i as required.

Now since $X_1u_i - (i+1)v_i \in J\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ for $1 \leq i \leq n-2$ and $\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ has the constant image property, there exist $s_i \in \mathfrak{K}(M)/J^2\mathfrak{K}(M)$ such that

$$X_1u_i - (i+1)v_i = X_1s_i.$$

Thus if $p \nmid (i+1)$, then $X_1\left(\frac{1}{i+1}(u_i - s_i)\right) = v_i$ so that

$$X_1X_2\left(\frac{1}{i+1}(u_i - s_i)\right) = X_2X_1\left(\frac{1}{i+1}(u_i - s_i)\right) = X_2v_i = w_{i+1}$$

and

$$X_2\left(\frac{1}{i+1}(u_i - s_i)\right) \equiv v_{i+1} \pmod{\text{Ker}(X_1, \mathfrak{K}(M)/J^2\mathfrak{K}(M))}.$$

Write $X_2\left(\frac{1}{i+1}(u_i - s_i)\right) = v_{i+1} + t_{i+1}$ with $t_{i+1} \in \text{Ker}(X_1, \mathfrak{K}(M)/J^2\mathfrak{K}(M))$. We claim that each $t_{i+1} \in J\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. If $p \nmid (i+1)$ and $p \nmid (i+2)$, then it follows from the equations (6.2) that

$$\begin{aligned} v_{i+1} + t_{i+1} &= X_2\left(\frac{1}{i+1}(u_i - s_i)\right) = \frac{1}{i+1}X_2u_i - \frac{1}{i+1}X_2s_i \\ &= \frac{1}{i+1}(X_1u_{i+1} - v_{i+1}) - \frac{1}{i+1}X_2s_i \\ &= \frac{1}{i+1}\left((i+2)X_1\frac{1}{i+2}(u_{i+1} - s_{i+1}) + X_1s_{i+1} - v_{i+1}\right) - \frac{1}{i+1}X_2s_i \\ &= \frac{1}{i+1}\left((i+2)v_{i+1} - v_{i+1} + X_1s_{i+1}\right) - \frac{1}{i+1}X_2s_i \\ &= v_{i+1} + \frac{1}{i+1}X_1s_{i+1} - \frac{1}{i+1}X_2s_i, \end{aligned}$$

hence $t_{i+1} \in J\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. On the other hand, if $p \mid (i+2)$, then $X_1u_{i+1} = X_1s_{i+1}$ and we have

$$v_{i+1} + t_{i+1} = \frac{1}{i+1}(X_1u_{i+1} - v_{i+1}) - \frac{1}{i+1}X_2s_i = \frac{1}{i+1}X_1s_{i+1} - \frac{1}{i+1}v_{i+1} - \frac{1}{i+1}X_2s_i.$$

The terms involving v_{i+1} then cancel so that $t_{i+1} \in J\mathfrak{K}(M)/J^2\mathfrak{K}(M)$.

The elements u'_i and v'_i may now be constructed inductively. For $p \mid i$, we set $v'_i = v_i$. Now assume by induction that v'_i has been constructed and that $v'_i = v_i + t'_i$ with $t'_i \in J\mathfrak{K}(M)/J^2\mathfrak{K}(M)$. Because $\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ has the constant image property, there exists $s'_i \in \mathfrak{K}(M)/J^2\mathfrak{K}(M)$ such that $X_1 s'_i = t'_i$. We then have

$$X_1 \left(\frac{1}{i+1}(u_i - s_i) + s'_i \right) = v_i + t'_i = v'_i.$$

So let $u'_{i+1} = \frac{1}{i+1}(u_i - s_i) + s'_i$. Then $X_2 u'_{i+1} = v_{i+1} + t_{i+1} + X_2 s'_i$, and the right two terms lie in $J\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ by the previous remarks. Setting $v'_{i+1} = v_{i+1} + t_{i+1} + X_2 s'_i$ now completes the induction. \square

APPENDIX A

Generic points in algebraic geometry

A.1. Generic points for affine varieties

In this section we recall some basic concepts from the Zariski-Weil theory of generic points for irreducible affine varieties. Let k be an algebraically closed field and let $U = \mathbb{A}^r(k)$ denote the r dimensional affine space over k . Viewing U simply as a k -vector space, we also consider the k -linear dual U^* and its symmetric algebra $S(U^*)$. It is well known that there is a 1-1 order reversing correspondence between the irreducible affine subvarieties of U and the prime ideals in $S(U^*)$, obtained in the following way: Fixing a basis x_1, \dots, x_r of U , we let y_1, \dots, y_r denote the respective dual basis of U^* , that is, each y_i is a k -linear map $U \rightarrow k$ such that $y_i(x_j) = \delta_{ij}$ for all $1 \leq j \leq r$. We then have $S(U^*) \cong k[y_1, \dots, y_r]$ as k -algebras, and under this isomorphism we can view any element of $S(U^*)$ as a polynomial function $U \rightarrow k$. The required correspondence then assigns to a prime ideal \mathfrak{p} in $S(U^*)$ the set

$$V(\mathfrak{p}) = \{u \in U \mid f(u) = 0 \text{ for all } f \in \mathfrak{p}\}.$$

It follows from an argument similar to that given in Proposition A.1 (below) that $V(\mathfrak{p})$ is independent of the choice of basis x_1, \dots, x_r of U .

Now let V be an irreducible closed subvariety of U corresponding to a prime ideal \mathfrak{p} in $S(U^*)$. The *coordinate ring* of V is defined to be the quotient algebra $k[V] = S(U^*)/\mathfrak{p}$. Since \mathfrak{p} is a prime ideal, $k[V]$ is an integral domain. We define the *function field* of V to be the field of fractions $k(V) = \text{foc}(k[V])$.

We now fix a basis x_1, \dots, x_r of U with dual basis y_1, \dots, y_r as above. Letting \bar{y}_i denote the image of y_i in $k[V]$, we define the *generic point* of V to be the element

$$\bar{y}_1 \otimes x_1 + \dots + \bar{y}_r \otimes x_r \in U \otimes_k k(V) \cong \mathbb{A}^r(k(V)).$$

Proposition A.1. *The generic point of V is independent of the choice of basis x_1, \dots, x_r of U .*

PROOF. Let x'_1, \dots, x'_r be another basis of U and let $A = (\lambda_{ij}) \in GL_r(k)$ be the matrix corresponding to this change of basis, i.e., set $x'_j = \sum_{i=1}^r \lambda_{ij} x_i$ for each $1 \leq j \leq r$. If y'_1, \dots, y'_r is the dual basis corresponding to x'_1, \dots, x'_r , then the matrix

corresponding to the change of basis y_1, \dots, y_r to y'_1, \dots, y'_r is given by A^{-1} . Letting $A^{-1} = (\mu_{ij})$, we thus have

$$\begin{aligned} \bar{y}'_1 \otimes x'_1 + \cdots + \bar{y}'_r \otimes x'_r &= \sum_{l=1}^r \left(\sum_{j=1}^r \mu_{lj} \bar{y}_j \otimes \sum_{i=1}^r \lambda_{il} x_i \right) = \sum_{i,j} \sum_{l=1}^r \lambda_{il} \mu_{lj} (\bar{y}_j \otimes x_i) \\ &= \sum_{i,j} \delta_{ij} (\bar{y}_j \otimes x_i) = \bar{y}_1 \otimes x_1 + \cdots + \bar{y}_r \otimes x_r. \end{aligned} \quad \square$$

If K/k is an algebraically closed field extension having transcendence degree at least r over k , then there exists an embedding $k(V) \hookrightarrow K$ since $k(V)$ has transcendence degree at most r over k and K is algebraically closed. Letting t_1, \dots, t_r denote the respective images of $\bar{y}_1, \dots, \bar{y}_r$ in K under such an embedding, we call the point

$$t_1 \otimes x_1 + \cdots + t_r \otimes x_r \in U \otimes_k K \cong \mathbb{A}^r(K)$$

a *generic point* of V .

Conversely, if K/k is a field extension, then any element $\alpha \in \mathbb{A}^r(K) \cong U \otimes_k K$ may be written in the form $t_1 \otimes x_1 + \cdots + t_r \otimes x_r$ with $t_1, \dots, t_r \in K$. If $\mathfrak{p} \subseteq k[y_1, \dots, y_r]$ is the set of polynomials annihilated by t_1, \dots, t_r , then \mathfrak{p} is a prime ideal, hence it corresponds to an irreducible subvariety V of U . The homomorphism $k(V) \rightarrow K$ given by $\bar{y}_i \mapsto t_i$ then induces an extension map $\mathbb{A}^r(k(V)) \rightarrow \mathbb{A}^r(K)$ under which the generic point of V is mapped to α . We then say that α is a generic point of V . Under this correspondence, we may view every point of $\mathbb{A}^r(K)$ as the generic point of some irreducible closed subvariety of U .

The following proposition unifies the above definitions of generic points and illustrates the concept of *generic properties*.

Proposition A.2. *If V is an irreducible closed subvariety of $\mathbb{A}^r(k)$ and T is any set of polynomials in $k[y_1, \dots, y_r]$, then the following are equivalent.*

- (i) *The generic point of V annihilates T .*
- (ii) *Any generic point of V defined over a field extension K/k annihilates T .*
- (iii) *Every point in V annihilates T .*

Moreover, if these conditions fail, then the points in V not annihilating T form a dense open subset of V .

PROOF. Let \mathfrak{p} be the prime ideal in $k[y_1, \dots, y_r]$ corresponding to V and let

$$\bar{y}_1 \otimes x_1 + \cdots + \bar{y}_r \otimes x_r \in \mathbb{A}^r(k(V))$$

be the generic point of V . If f is any polynomial in $k[y_1, \dots, y_r]$, then we may view f as a polynomial function $\mathbb{A}^r(k(V)) \rightarrow k(V)$ by expanding the relation

$$y_i(\bar{y}_j \otimes x_l) = \bar{y}_j \delta_{il} \quad \text{for all } 1 \leq i, j, l \leq r$$

using the distributive law. Under this assignment,

$$f(\bar{y}_1 \otimes x_1 + \dots + \bar{y}_r \otimes x_r)$$

is simply the coset in $k[y_1, \dots, y_r]/\mathfrak{p}$ containing f , thus the generic point of V annihilates f if and only if $f \in \mathfrak{p}$. This establishes (i) \Leftrightarrow (iii).

Now let α be a generic point of V defined over some field extension K/k as described above and let ι denote the injection $k(V) \hookrightarrow K$ under which the induced map

$$\tilde{\iota} : \mathbb{A}^r(k(V)) \rightarrow \mathbb{A}^r(K)$$

sends $\bar{y}_1 \otimes x_1 + \dots + \bar{y}_r \otimes x_r$ to α . If f is any polynomial in $k[y_1, \dots, y_r]$, then

$$\begin{aligned} f(\alpha) &= f(\tilde{\iota}(\bar{y}_1 \otimes x_1 + \dots + \bar{y}_r \otimes x_r)) \\ &= \iota(f(\bar{y}_1 \otimes x_1 + \dots + \bar{y}_r \otimes x_r)). \end{aligned}$$

Hence α annihilates f if and only if $\bar{y}_1 \otimes x_1 + \dots + \bar{y}_r \otimes x_r$ does, and this establishes (i) \Leftrightarrow (ii).

Finally, the set of points in V that do not annihilate some element of T are precisely those in $V \setminus V(T)$, which is clearly an open subset of V in the Zariski topology. If this subset is non-empty, then it is also dense in V since V is irreducible. \square

A.2. Generic points in the language of schemes

The theory of generic points for affine varieties can be extended to the theory of schemes over a fixed field k with the appropriate modifications. If $X \rightarrow \text{Spec } k$ is a scheme over k and x is any point in X , then there exists an affine open neighbourhood $U \cong \text{Spec } A$ of x on which x is represented by a prime ideal \mathfrak{p} of A . It follows from [29, Corollaire 10.4.7] that if X is locally of finite type over k , then x is closed in X if and only if \mathfrak{p} is a maximal ideal in A . Recall that the *residue field* of x is defined to be the quotient $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, where \mathfrak{m}_x is the unique maximal ideal in the local ring $\mathcal{O}_{X,x}$. As outlined in [30, Section II.3], we may form the fibred product

$$X_{k(x)} = X \times_{\text{Spec } k} \text{Spec } k(x).$$

Note that $X_{k(x)}$ contains the affine open subset

$$U_{k(x)} \cong U_i \times_{\text{Spec } k} \text{Spec } k(x) \cong \text{Spec } A \otimes_k k(x).$$

Letting \mathfrak{m} be the ideal of $A \otimes_k k(x) \cong A \otimes_k A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ generated by the elements $a \otimes 1 - 1 \otimes \bar{a}$ for all $a \in A$, we have

$$(A \otimes_k k(x))/\mathfrak{m} \cong k(x),$$

hence \mathfrak{m} is a maximal ideal of $A \otimes_k k(x)$. Moreover, the image of \mathfrak{m} under the projection $U_{k(x)} \rightarrow U$ is \mathfrak{p} . It follows that if y is the closed point in $X_{k(x)}$ represented by \mathfrak{m} on $U_{k(x)}$, then the image of y under the projection $X_{k(x)} \rightarrow X$ is x . We call the closed point $y \in X_{k(x)}$ defined in this way *the generic point* of x .

Similarly, if K/k is any field extension and α is a closed point in X_K with $K(\alpha) = K$, then we call α a *generic point* for its image $x \in X$ under the projection $X_K \rightarrow X$. The latter is a scheme morphism mapping $\alpha \mapsto x$, thus there exists a corresponding non-zero (hence injective) field homomorphism $k(x) \rightarrow K(\alpha) = K$. Observe that the image of α under the projection

$$X_K \cong X \times_{\text{Spec } k} \text{Spec } k(x) \times_{\text{Spec } k(x)} \text{Spec } K \longrightarrow X \times_{\text{Spec } k} \text{Spec } k(x) = X_{k(x)}$$

necessarily maps α to the generic point of x in $X_{k(x)}$.

APPENDIX B

Modules for \mathbb{Z}/p

Let p be a prime number and k an algebraically closed field of characteristic p . In this appendix we briefly outline the classification of finite dimensional $k(\mathbb{Z}/p)$ -modules in terms of their Jordan types. This theory can be thought of as the primary motivation for studying modules over elementary abelian p -groups in terms of their restrictions to cyclic shifted subgroups. (See Section 2.2.)

Let $\langle g \rangle \cong \mathbb{Z}/p$ be a cyclic group of order p and define $X = g - 1 \in J(k\langle g \rangle)$. For a finite dimensional $k\langle g \rangle$ -module M , let $\rho_M : k\langle g \rangle \rightarrow \text{Mat}_{\dim_k(M)}(k)$ be the matrix representation affording M . If M is indecomposable, then since $X^p = 0$, the Jordan canonical form $\rho_M(X)'$ of $\rho_M(X)$ contains only one Jordan block of length $\dim_k(M)$ with eigenvalue zero. In particular, we may conclude that $\dim_k(M) \leq p$. The matrix representation defined by $g \mapsto I_{\dim_k(M)} + \rho_M(X)'$ is then a representation of $k\langle g \rangle$ equivalent to ρ_M , having been obtained from ρ_M via conjugation by an element of $GL_{\dim_k(M)}(k)$. Conversely, if $1 \leq i \leq p$ and $J_i \in \text{Mat}_i(k)$ is the Jordan block of length i with eigenvalue zero, then the map $g \mapsto I_i + J_i$ defines an indecomposable matrix representation $k\langle g \rangle \rightarrow \text{Mat}_i(k)$. This establishes a 1-1 correspondence between isomorphism types of indecomposable $k\langle g \rangle$ -modules and Jordan blocks having length $1 \leq i \leq p$ over k and eigenvalue zero. With a slight abuse of notation, we shall also denote the indecomposable representation of dimension $1 \leq i \leq p$ by J_i .

As in Section 2.2, we denote by $M \tilde{\otimes}_k N$ the vector space $M \otimes_k N$ endowed with the $k\langle g \rangle$ -module structure induced by the Lie theoretic comultiplication

$$X \mapsto X \otimes 1 + 1 \otimes X,$$

reserving the notation $M \otimes_k N$ for the $k\langle g \rangle$ -module defined via the group theoretic comultiplication $g \mapsto g \otimes g$. Our immediate goal is to prove that these two structures are isomorphic for cyclic groups of order p .

Lemma B.1. *If $1 \leq i < p$, then $J_2 \otimes_k J_i \cong J_2 \tilde{\otimes}_k J_i \cong J_{i-1} \oplus J_{i+1}$. We also have $J_2 \otimes_k J_p \cong J_2 \tilde{\otimes}_k J_p \cong J_p \oplus J_p$.*

PROOF. For $J_2 \otimes_k J_i$ we have

$$\rho_{J_2 \otimes_k J_i}(g) = (I_2 + J_2) \otimes (I_i + J_i) = \begin{pmatrix} I_i + J_i & I_i + J_i \\ 0 & I_i + J_i \end{pmatrix}$$

so that

$$\rho_{J_2 \otimes_k J_i}(X) = \rho_{J_2 \otimes_k J_i}(g) - I_{2i} = \begin{pmatrix} J_i & I_i + J_i \\ 0 & J_i \end{pmatrix}.$$

Recall that the product of 2×2 block matrices is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}.$$

It follows that

$$\rho_{J_2 \otimes_k J_i}(X^n) = \begin{pmatrix} J_i & I_i + J_i \\ 0 & J_i \end{pmatrix}^n = \begin{pmatrix} J_i^n & n(J_i^{n-1} + J_i^n) \\ 0 & J_i^n \end{pmatrix}.$$

Similarly, for $J_2 \tilde{\otimes}_k J_i$ we have

$$\rho_{J_2 \tilde{\otimes}_k J_i}(X) = J_2 \otimes I_i + I_2 \otimes J_i = \begin{pmatrix} J_i & I_i \\ 0 & J_i \end{pmatrix}$$

so that

$$\rho_{J_2 \tilde{\otimes}_k J_i}(X^n) = \begin{pmatrix} J_i & I_i \\ 0 & J_i \end{pmatrix}^n = \begin{pmatrix} J_i^n & nJ_i^{n-1} \\ 0 & J_i^n \end{pmatrix}.$$

From this we see that

$$\text{rank}(X^n, J_2 \otimes_k J_i) = \text{rank}(X^n, J_2 \tilde{\otimes}_k J_i) = \begin{cases} 2i - 2n & n < i \\ 1 & n = i < p \\ 0 & n = i = p. \end{cases}$$

The proof now follows from the fact that the isomorphism types of $J_2 \otimes_k J_i$ and $J_2 \tilde{\otimes}_k J_i$ are determined by the Jordan types of $\rho_{J_2 \otimes_k J_i}(X)$ and $\rho_{J_2 \tilde{\otimes}_k J_i}(X)$, respectively, and the Jordan type of a nilpotent matrix is completely determined by the ranks of its powers. \square

Recall that the *representation ring* of $k(\mathbb{Z}/p)$ is the ring $a(k(\mathbb{Z}/p)) = a(\mathbb{Z}/p)$ whose elements are the isomorphism classes $[M]$ of finite dimensional $k(\mathbb{Z}/p)$ -modules, with addition and multiplication defined by

$$[M] + [N] = [M \oplus N], \quad [M][N] = [M \otimes_k N].$$

We also define the *Lie theoretic representation ring* $\tilde{a}(\mathbb{Z}/p)$ to be the ring having the same additive structure as $a(\mathbb{Z}/p)$, but with multiplication given via the Lie theoretic tensor product $[M][N] = [M \tilde{\otimes}_k N]$.

Lemma B.2. *For p odd, the isomorphism class $[J_2]$ generates $a(\mathbb{Z}/p)$ and $\tilde{a}(\mathbb{Z}/p)$.*

PROOF. We first prove the statement for $a(\mathbb{Z}/p)$. For this it suffices to show that $[J_2]$ generates $[J_1] = [k]$. Note by Lemma B.1 that

$$[J_2][J_i] = [J_2 \otimes_k J_i] = [J_{i-1} \oplus J_{i+1}] = [J_{i-1}] + [J_{i+1}]$$

for all $2 \leq i \leq p-1$ and $[J_2][J_p] = [J_2 \otimes_k J_p] = [J_p \oplus J_p] = 2[J_p]$. It follows that

$$[J_1] = [J_2]([J_2] - [J_4] + [J_6] - \cdots \pm [J_{p-1}] \mp \frac{1}{2}[J_p]).$$

A similar argument shows that $[J_2]$ generates $\tilde{a}(\mathbb{Z}/p)$. □

Proposition B.3. *$a(\mathbb{Z}/p)$ and $\tilde{a}(\mathbb{Z}/p)$ have equal ring structures.*

PROOF. The case $p = 2$ is obvious, and the proof for $p > 2$ follows directly from Lemmas B.1 and B.2. □

Corollary B.4. *If M and N are finite dimensional $k(\mathbb{Z}/p)$ -modules, then*

$$M \otimes_k N \cong M \tilde{\otimes}_k N.$$

APPENDIX C

A brief introduction to rank varieties

In this appendix we recall the definition of the rank variety of a finite dimensional kE -module. This material will be used in Section 2.8 to show that the rank variety is too coarse of an invariant to yield useful information about kE -modules of constant Jordan type. A standard reference for the following is Section 5.8 of Benson [4].

Studying kE -modules in terms of their restrictions to cyclic shifted subgroups was pioneered by Dade. The following theorem, which appeared as Lemma 11.8 of [21], is the starting point of our discussion. Recall that since kE is a local ring, a kE -module is projective if and only if it is free.

Theorem C.1 (Dade's lemma). *A finite dimensional kE -module M is projective if and only if $M \downarrow_{\langle g_\alpha \rangle}$ is projective for all non-zero $\alpha \in \mathbb{A}^r(k)$.*

If $\langle g \rangle \cong \mathbb{Z}/p$ is a cyclic group of order p , then a $k\langle g \rangle$ -module M is projective if and only if the Jordan type of $X = g - 1$ on M contains only blocks of length p . In this case, writing $M \cong (k\langle g \rangle)^s$, we have $\text{rank}(X, M) = s(p - 1)$. This shows that if M is a kE -module and α is a non-zero point in $\mathbb{A}^r(k)$, then $M \downarrow_{\langle g_\alpha \rangle}$ is projective if and only if p divides $\dim_k(M)$ and the rank of X_α on M is $(p - 1) \dim_k(M)/p$. Denoting the Jordan type of X_α on M by $[p]^{a_p} \dots [1]^{a_1}$, we also have

$$\begin{aligned} \text{rank}(X_\alpha, M) &= \sum_{j=1}^p (j-1)a_j = \dim_k(M) - \sum_{j=1}^p a_j \leq \dim_k(M) - \left(\left\lfloor \frac{\dim_k(M)}{p} \right\rfloor + 1 \right) \\ &\leq \dim_k(M) - \frac{\dim_k(M)}{p} = \frac{(p-1)\dim_k(M)}{p} \end{aligned}$$

since $\sum_{j=1}^p a_j$ is the number of Jordan blocks in the action of X_α on M , and each such block has length at most p . It follows that $M \downarrow_{\langle g_\alpha \rangle}$ is *not* projective if and only if the rank of X_α on M is strictly less than $(p - 1) \dim_k(M)/p$. As observed in Section 2.1, the points $\alpha \in \mathbb{A}^r(k)$ satisfying the latter condition form a Zariski closed subset of $\mathbb{A}^r(k)$. This justifies the following definition.

Definition C.2 (Carlson [12]). If M is a kE -module, then the *rank variety* of M is the homogeneous subvariety

$$V_E^\sharp(M) = \{0 \neq \alpha \in \mathbb{A}^r(k) \mid M \downarrow_{\langle g_\alpha \rangle} \text{ is not projective}\} \cup \{0\}$$

of $\mathbb{A}^r(k)$.

Combining the above definition with Dade's lemma, we immediately obtain the following characterisation of projectivity.

Corollary C.3. *A kE -module M is projective if and only if $V_E^{\sharp}(M) = \{0\}$.*

Bibliography

- [1] M. F. Atiyah and I. G. MacDonald. *Introduction to commutative algebra*. Addison-Wesley, 1969.
- [2] G. S. Avrunin and L. L. Scott. Quillen stratification for modules. *Invent. Math.*, 66:277–286, 1982.
- [3] D. J. Benson. *Representations and Cohomology I: Basic representation theory of finite groups and associative algebras*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1991. Reprinted in paperback, 1998.
- [4] D. J. Benson. *Representations and Cohomology II: Cohomology of groups and modules*, volume 31 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1991. Reprinted in paperback, 1998.
- [5] D. J. Benson. Modules of constant Jordan type with one non-projective block. *Algebras and Representation Theory*, 13:315–318, 2010.
- [6] D. J. Benson. Modules of constant Jordan type and a conjecture of Rickard. To appear in *J. Algebra*, 2012.
- [7] D. J. Benson. Modules of constant Jordan type with small non-projective part. To appear in *Algebras and Representation Theory*, 2012.
- [8] D. J. Benson and J. F. Carlson. Diagrammatic methods for modular representations and cohomology. *Commun. in Algebra*, 15:53–121, 1987.
- [9] D. J. Benson and J. Pevtsova. A realization theorem for modules of constant Jordan type and vector bundles. *Trans. Amer. Math. Soc.*, 364:6459–6478, 2012.
- [10] V. M. Bondarenko and Y. A. Drozd. Representation type of finite groups. (*Transl. from Russian*) *J. Soviet Math.*, 20:2515–2528, 1982.
- [11] T. Bühler. Exact categories. *Expo. Math.*, 28:1–69, 2010.
- [12] J. F. Carlson. The complexity and varieties of modules. In *Integral representations and their applications, Oberwolfach, 1980*, volume 882 of *Lecture Notes in Mathematics*, pages 415–422. Springer-Verlag, Berlin/New York, 1981.
- [13] J. F. Carlson. The varieties and cohomology ring of a module. *J. Algebra*, 85:104–143, 1983.
- [14] J. F. Carlson. *Modules and Group Algebras*. Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag, Basel, 1996.
- [15] J. F. Carlson and E. M. Friedlander. Exact category of modules of constant Jordan type. In *Algebra, arithmetic and geometry: Manin Festschrift*, volume 269 of *Progr. in Math.*, pages 259–281. Birkhäuser Verlag, Basel, 2009.
- [16] J. F. Carlson, E. M. Friedlander, and J. Pevtsova. Modules of constant Jordan type. *J. Reine & Angew. Math.*, 614:191–234, 2008.
- [17] J. F. Carlson, E. M. Friedlander, and A. A. Suslin. Modules for $\mathbb{Z}/p \times \mathbb{Z}/p$. *Comment. Math. Helvetici*, 86:609–657, 2011.

- [18] L. Chouinard. Projectivity and relative projectivity over group rings. *J. Pure & Applied Algebra*, 7:278–302, 1976.
- [19] D. Collingwood and M. McGovern. *Nilpotent orbits in semisimple Lie algebras*, volume 43. Van Nostrand Reinhold, New York, 1993.
- [20] E. C. Dade. Endo-permutation modules over p -groups, I. *Ann. of Math.*, 107:459–494, 1978.
- [21] E. C. Dade. Endo-permutation modules over p -groups, II. *Ann. of Math.*, 108:317–346, 1978.
- [22] L. Dornhoff. *Group representation theory part A: Ordinary representation theory*. Dekker, New York, 1971.
- [23] L. Dornhoff. *Group representation theory part B: Modular representation theory*. Dekker, New York, 1972.
- [24] Y. A. Drozd. Tame and wild matrix problems. In *Representation Theory II*, volume 832 of *Lecture Notes in Mathematics*, pages 242–258. Springer-Verlag, Berlin/New York, 1980.
- [25] E. M. Friedlander and J. Pevtsova. Constructions for infinitesimal group schemes. *Trans. Amer. Math. Soc.*, 363:6007–6061, 2011.
- [26] E. M. Friedlander, J. Pevtsova, and A. Suslin. Generic and maximal Jordan types. *Invent. Math.*, 168:485–522, 2007.
- [27] W. Fulton. *Intersection theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Folge 3, Band 2. Springer-Verlag, Berlin/New York, 1984.
- [28] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.
- [29] A. Grothendieck and J. Dieudonné. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Publ. Math. Inst. Hautes Études Sci.*, 28, 1966.
- [30] R. Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin/New York, 1977.
- [31] J. C. Jantzen. *Representations of algebraic groups*. Academic Press, New York/London, 1987.
- [32] C. Okonek, M. Schneider, and H. Spindler. *Vector bundles on complex projective spaces*. Birkhäuser Verlag, Basel, 1980; 3ed. 2011.
- [33] D. G. Quillen. The spectrum of an equivariant cohomology ring, I. *Ann. of Math.*, 94:549–572, 1971.
- [34] D. G. Quillen. The spectrum of an equivariant cohomology ring, II. *Ann. of Math.*, 94:573–602, 1971.
- [35] W. Wheeler. Generic module theory. *J. Algebra*, 185:205–228, 1996.